AN EASY

INTRODUCTION

TO THE

THEORY and PRACTICE

OF

MECHANICS;

Containing a VARIETY of

CURIOUS and IMPORTANT PROBLEMS

Inveſtigated with the greateſt Facility by the

APPLICATION of One GENERAL PROPERTY

OF THE

CENTER OF GRAVITY

Without having RECOURSE to the

COMPOSITION and RESOLUTION of FORCES.

By SAMUEL CLARK.

LONDON:

Printed for J. NOURSE, oppoſite *Katherine-Street* in the *Strand*,
Bookſeller in Ordinary to His MAJESTY.

M DCC LXIV.

43.
3 9
87

TO

WILLIAM FELLOWS,

OF

Shotteſham, in the County of *Norfolk*, Eſq;

SIR,

THE Honour of having been admitted to Your Acquaintance, I regard as the moſt favourable Incident of my Life, eſpecially as it now affords me an Opportunity of publickly acknowledging my Obligations to Mr. FELLOWS; which indeed are of ſuch a Nature, that to omit it, would be the higheſt Injuſtice, as well as Ingratitude. I know very well, that the juſt Tribute I now pay You, is offering a ſort of Violence to One, who is as ſoli-

ſolicitous to ſhun Applauſe, as He is careful to deſerve it. It is for this Reaſon, and the Impoſſibility of doing Juſtice to Your Character, without being ſuſpected of that Flattery ſo peculiar to *Dedications*, that I forbear to enumerate thoſe extraordinary Qualities by which You are ſo eminently diſtinguiſhed, and only beg Leave to aſſure You, that I am, with the greateſt Reſpect,

SIR,

Your moſt obliged,

Humble Servant,

SAMUEL CLARK.

THE

PREFACE.

THE motives that induced me to publiſh the following ſheets, were chiefly to ſettle the buſineſs of Mechanics upon the moſt certain and undeniable principles, and to point out and provide againſt ſuch material errors, as were moſt likely to ariſe from a too general application of the method of dividing forces; which notwithſtanding may, in many caſes, produce juſt concluſions, yet there are others wherein that method will prove defective. I have, in the enſuing pages, deduced the ſolutions to a great variety of important problems, by help of one general property of the center of gravity of a body, or of the common center of gravity of a ſyſtem of bodies; to wit, that thoſe centers will, when the bodies are at reſt, be in the loweſt place poſſible. The former of theſe caſes has been univerſally aſſented to; and the latter is, I think, ſo very obvious, that any attempt towards a demonſtration thereof, would only ſerve to render it leſs evident; eſpecially when we conſider, that as the abſolute motion of a ſyſtem of bodies acting mutually upon each other, depends entirely upon the motion of their common center of gravity; and if, into this point or center, the whole ſyſtem were contracted, its motion would remain uninfluenced by ſuch a change made in the diſpoſition of the parts, it muſt, as bodies can only move but to deſcend, neceſſarily follow, that, when the common center of gravity of a ſyſtem of bodies is in the loweſt place, that ſyſtem can have no tendency to move, becauſe, in any other poſition, the deſcent of the center of gravity of the bodies would (by the hypotheſis) be leſs than before.

THERE cannot, I apprehend, remain any doubt with regard to the certainty of the principle, upon which the inveſtigations in the enſuing Work are founded; for in a great number of examples, where the reſolution of forces can with propriety be applied, the concluſions by either method are exactly the ſame. This may indeed ſerve to ſatisfy thoſe, who are ſo tenacious of the method of treating mechanic problems by the diviſion and compoſition of forces, as ſcarcely to admit of any other; yet I flatter myſelf, that the method I have purſued will, to the impartial reader, appear ſo ſatisfactory, as to remain a future criterion in mechanic diſquiſitions.

THE inveſtigations with regard to the equilibrium of beams, ſuſtained by means of ſtrings, are very different to thoſe required for the ſame purpoſe, when the beams are ſuſtained by weights; or, which is the ſame thing, by forces acting in the directions of thoſe ſtrings, to whoſe ends the weights were appended. For, when a beam is ſuſtained at reſt by means of two ſtrings, it is very certain, the center of gravity of that beam will be in the loweſt place; but, on the other hand, when two weights, acting in the former directions of the ſtrings, perform the office of ſuſtaining the beam, it is the common center of gravity of the beam and weights that muſt poſſeſs the loweſt place; and as there are innumerable caſes, wherein thoſe centers may not coincide, or where the deſcent of one may be a maximum when that of the other is not, it follows, that a change of ſituation may enſue, under the different circumſtances of ſtrings and weights: and to the want of having properly attended to this particular, I apprehend, may be aſcribed the miſtakes which ſome conſiderable writers on Mechanics have fell into; becauſe the ſame method of reaſoning, by which they have deduced the true properties of the lever, wedge, ſcrew, *&c.* and alſo of bodies ſuſtained upon inclined planes, cannot be extended to all caſes of ſuſpended beams. To inſtance this, the reader is deſired to refer to *page* 27, where, in a ſolution to the Xth Problem, according to Mr. *Simpſon*'s method, tranſcribed from his Treatiſe on Fluxions, it appears, that the tenſion of two cords AD, BC, (ſee *fig.* on *p.* 26.) ſuſtaining a beam or rod CD, at its extremes D and C, are to each other as ſin BCD ×

CG to ſin ADC × DG reſpectively; but, by the ſolution to Problem XXVIII, thoſe tenſions are to each other as $\frac{AD}{BC}$ to $\frac{AF}{HB}$. This latter ratio of the tenſions ſhews, that the beam CD cannot be ſupported in that poſition, wherein ABC is a right angle, becauſe HB then vaniſhing, the ratio of Q to P becomes infinite; but the former points out no ſuch reſtriction with regard to the angle ABC, for as the ratio of the tenſions there depends upon the meaſure of the angles ADC, BCD, which always remain finite, be the angle ABC what it will, there can no poſition of the beam be aſſigned, wherein that ratio cannot be aſcertained. Another proof of the inſufficiency of the abovementioned ratio of the tenſions of the cords may be deduced from that well known principle in Mechanics, mentioned at *page* 33, whereby it appears, that ABC being a right angle, if AD and HC be produced till they interſect, a right line drawn from the point of interſection, through the center of gravity G of the beam, ſhould be perpendicular to the horizontal line AB; but as this is manifeſtly impoſſible, when BC coincides with CH, or, which is the ſame thing, when ABC is a right angle, it follows, that no forces whatever, acting in the directions AD and BC, can ſuſtain the beam DC in the propoſed ſituation. The fault here ſeems to ariſe from ſuppoſing the forces, acting in the directions AD and BC, to continue the ſame when the beam is freely ſuſpended as they were before, when the ends C and D were moveable about thoſe points by means of pins fixed therein alternately; for although it is very certain, that Q is equal to $\frac{\text{ſin FDC} \times \text{CG}}{\text{ſin ADC} \times \text{CD}} \times w$, when DC is moveable about the fixed point C; and alſo, that $\frac{\text{ſin HDC} \times \text{DG}}{\text{ſin BCD} \times \text{DC}} \times w$, will expreſs the weight neceſſary to ſuſtain the beam CD, when moveable about the point D as fixed; yet when the beam is freely ſuſpended by the weights Q and P, acting thereon at the ſame time, their reſpective values, and conſequently their ratio, will be changed from what they were before: this will appear evident by the ſolution to the XXVIIIth Problem.

IT is ſomething remarkable, that in the inveſtigations relating to the ſolutions of mechanic problems, it often happens the inferior caſes are not included. Inſtances of this kind occur in ſeveral of the problems before us, as at *page* 55; wherein, if we ſuppoſe ED = o, the problem there propoſed will become exactly the ſame with Problem I. Now when ED = o, the firſt general equation in the ſolution to the XXVIIth Problem will be reduced to $\frac{Q}{AE} = \frac{W}{DH}$, and the ſecond will entirely vaniſh; AE and DH in the *figure* at *page* 55, correſpond with AC and EC reſpectively, in the *figure* to Problem I; therefore $\frac{Q}{AC}$ ſhould be equal to $\frac{W}{EC}$, and conſequently Q : W :: AC : EC; but, by the ſolution to the Iſt Problem, Q : W :: $\frac{EB}{EC}$: $\frac{AB}{AC}$, whence it is evident, that the general ſolution to the XXVIIth Problem, does not include the particular caſe propoſed. In the very ſame manner it may be ſhewn, that the equation $\frac{D \times AE}{AC} = \frac{F \times BE}{BC}$ at *page* 9, cannot be deduced from either of the equations at *page* 57, by making ED = o, notwithſtanding that when ED = o, the problem is exactly the ſame with that at *page* 8.

AN

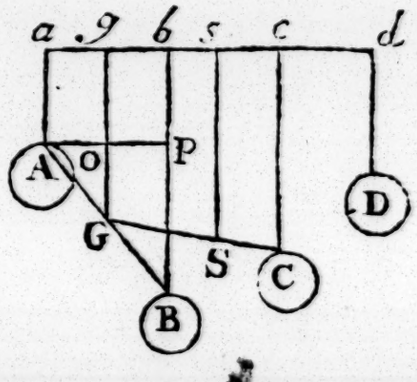
AN EASY
INTRODUCTION
TO THE
THEORY *and* PRACTICE
OF
MECHANICS.

IN order to facilitate the operations for determining the place of the center of gravity of a body, or that of the common center of gravity of a ſyſtem of bodies, (the chief difficulties that occur in the enſuing pages) the two following lemmas are premiſed.

LEMMA I.

LET *a c d* be a horizontal line, to which the weights A B C D, &c. are appended by means of the ſtrings A *a*, B *b*, C *c*, D *d*, &c. then will (M) the perpendicular diſtance of the common center of gravity of thoſe weights from the ſaid horizontal line, be equal to the quotient ariſing by dividing the ſum of the products of the weights into the lengths of their reſpective ſtrings, by the ſum of all the weights; that is,

$$M=\frac{A\,a\times A+B\,b\times B+C\,c\times C+D\,d\times D,\,\&c.}{A+B+C+D,\,\&c.}$$

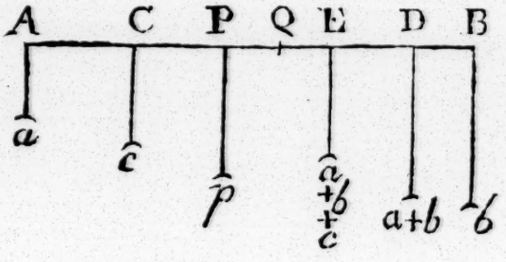


join the points A, B, with a right line, and let G be the place of the common center of gravity of the weights A, B, draw G *g*, and A P reſpectively parallel to A *a*, and *a c d*. Then by the ſimilar triangles A P B A O G, we have A B : A G :: P B : G O, whence $\frac{AG}{AB}\times B\,b-A\,a+A\,a$, or $\frac{AG}{AB}\times B\,b+\frac{BG}{AB}\times A\,a=G\,g$, but by the property of the center of gravity A G × A = B G × B; therefore A G : B G :: B : A, and A B : A G :: A + B : B, alſo A B : B G :: A + B : A, from theſe proportions we get

$\frac{AG}{AB}=\frac{B}{A+B}$ and $\frac{BG}{AB}=\frac{A}{A+B}$, let theſe be put for their equals in the laſt found value of Gg, and we have $\frac{B}{A+B}\times bB+\frac{A}{A+B}\times bB$, or $\frac{A\times Aa+B\times Bb}{A+B}=Gg$. Let the weights A, B, whoſe ſum call W, be placed at G, their common center of gravity; then if we join the points G C with a right line, and ſuppoſe S to be the common center of gravity of the weights W, C, the perpendicular diſtance Ss of that center from the horizontal line will, by the ſame way of reaſoning as before, be expreſſed by $\frac{Gg\times W+Cc\times C}{W+C}$ in which for Gg and W, ſubſtitute their reſpective values, *viz.* $\frac{A\times Aa+B\times Bb}{A+B}$ and A+B, we have $Ss=\frac{A\times Aa+B\times Bb+C\times Cc}{A+B+C}$

In the ſame manner it may be proved, that the perpendicular diſtance of the common center of gravity of the four weights A, B, C, D, from the horizontal line $a\,c\,d$, is equal to $\frac{A\times Aa+B\times Bb+C\times Cc+D\times Dd}{A+B+C+D}$ whence the law of continuation becomes manifeſt.

As this operation only determines the neareſt diſtance of that line (drawn parallel to the horizon) from $a\,d$, in which the common center of gravity of the propoſed weights will always be found, we muſt, if the real place of that center be required, alſo inveſtigate the point in $a\,d$, from whence a perpendicular being let fall, will, by its interſection with the above-mentioned parallel, give the place of the common center of gravity of all the weights. In order to which let a, b, be two weights hanging at the rod A B, and ſuppoſe D the place of their common center of gravity, then we have $AD\times a=DB\times b$, whence $AD:DB::b:a$, and $AD+DB:AD::a+b:b$, that is $AB:AD::b+a:b$, therefore $AD=\frac{AB\times b}{a+b}$.



Again, let c be another weight appended at C, and let them all be transferred to E, then to find this point we have $CE\times c=ED\times\overline{a+b}$ $\cdot\cdot$ $EC:ED::a+b:c$, and $EC+ED:ED::a+b+c:c$, that is $CD:ED::a+b+c:c$, and $ED=\frac{CD\times c}{a+b+c}$. $AE=AD-ED=\frac{AB\times b}{a+b}-\frac{CD\times c}{a+b+c}$; but $AD-AC=CD$, and $CD\times c=\frac{AB\times bc}{a+b}$

$-AC\times c=\frac{AB\times bc-AC\times cb-AC\times ca}{a+b}$, therefore $AE=\frac{AB\times b}{a+b}$

$-\frac{AB\times bc-AC\times bc-AC\times ac}{\overline{a+b}\times\overline{a+b+c}}$, equal to $\frac{AB\times b\times\overline{a+b+c}-AB\times bc+AC\times cb}{\overline{b+a}\times\overline{a+b+c}}$

$\frac{+AC\times ac}{\overline{b+a}\times\overline{a+b+c}}=\frac{AB\times b+AC\times c}{a+b+c}$. Let p be another weight beſides the former, all ſuſpended in equilibrio in the point Q. Then $PQ\times p=QE\times\overline{a+b+c}\cdot\cdot\cdot PQ:QE::a+b+c:p$, and $PE:QE::a+b+c+p:p$, whence $QE=\frac{PE\times p}{a+b+c+p}$, $AQ=AE-QE=\frac{AB\times b+AC\times c}{a+b+c}-\frac{PE\times p}{a+b+c+p}$, but $AE-AP=PE$, and $PE\times p=\frac{BA\times bp+AC\times cp}{a+b+c}-AP\times p$, whence $AQ=\frac{AB\times b+AC\times c}{a+b+c}-\frac{BA\times bp-AC\times pc+AP\times p\times\overline{a+b+c}}{\overline{a+b+c}\times\overline{a+b+c+p}}=\frac{\overline{AB\times b+AC\times c}\times p\times\overline{a+b+c}}{\overline{a+b+c}\times\overline{a+b+c+p}}$

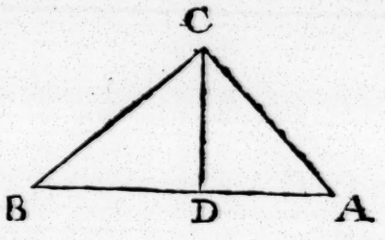
$\frac{-\overline{BA\times b-AC\times c}\times p+AP\times p\times\overline{a+b+c}}{\overline{a+b+c}\times\overline{a+b+c+p}}$ equal to $\frac{AB\times b+AC\times c+AP\times p}{a+b+c+p}$.

From theſe operations it appears, that the ſum of the products of the weights by their reſpective diſtances from A (ſuppoſed the point of ſuſpenſion) divided by the ſum of all the weights, gives the diſtance of their common center of gravity from the point A.

This property of the center of gravity of a ſyſtem of bodies is demonſtrated by moſt writers on Mechanics, but in a manner different to the above.

LEMMA II.

FROM the ſum of the ſquares of any two ſides of a plane triangle, ſubtract the double product of thoſe two ſides into the coſine (radius being unity) of the angle they contain, the remainder will give the ſquare of the third ſide.



Let ABC be any plane triangle, from an angular point C let fall upon AB the perpendicular CD, it is obvious that $2AB \times BD = AB \times BD + AB \times BD$, but $AB - AD = DB$, therefore $2AB \times BD = AB^2 - AB \times AD + AB \times BD$, or $AB \times \overline{AD - DB} = AB^2 - 2AB \times BD$, that is $AB^2 - 2AB \times BD = AD^2 - DB^2$, but $BC^2 - BD^2 = AC^2 - AD^2$ by the figure, whence by adding theſe two laſt equations we get $AB^2 + BC^2 - 2AB \times BD = AC^2$, or which is the ſame thing, $AB^2 + BC^2 - 2AB \times BC \times \frac{BD}{BC} = AC^2$. Now if BC be radius, BD will be the coſine of the angle ABC, but to the radius 1 $\frac{BD}{BC}$ is that coſine, therefore $AB^2 + BC^2 - 2AB \times BC \times \cos < ABC = AC^2$.

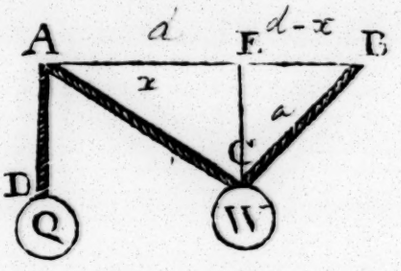
If the included angle ABC be obtuſe, then becauſe the coſine of an angle greater than 90°, is always denoted by that of its ſupplement with a negative ſign the laſt equation will be changed into $AB^2 + BC^2 + 2AB \times BC \times \cos < ABC = AC^2$. When ABC becomes a right angle, $2AB \times BC \times \cos < ABC$ vaniſhes, and $AB^2 + BC^2$ is then equal to AC^2, a well known property of a right angled plane triangle. The demonſtration of this lemma may be eaſily obtained by help of the 12th and 13th propoſitions of the ſecond book of *Euclid*'s Elements, for in the 13th it is proved, that in an acute angled plane triangle ABC, $AB^2 + BC^2 = AC^2 + 2AB \times DB$; and in the 12th, that, when ABC is an obtuſe angle, $AB^2 + BC^2 = AC^2 - 2AB \times DB$, in either caſe, by making radius equal to unity, and ſubtracting as before, the very ſame theorems will be produced.

PROBLEM

PROBLEM I.

TO determine the position of the thread BCAD (having one end fastened to a tack at B, the other end passing freely over a small pulley at A in the horizontal line, BA) when two known weights Q, w, appended at the given points D, C, sustain each other in equilibrio.

SOLUTION.



Put AB$=d$, BC$=a$, CAD$=l$, draw CE perpendicular to AB, and call AE, x, then EB$=d-x$, whence EC$=\sqrt{a^2-d^2+2dx-x^2}$ and AC$=\sqrt{a^2-d^2+2dx}$, therefore AD$=l-\sqrt{a^2-d^2+2dx}$, and (by Lemma 1.) the perpendicular distance of the common center of gravity of the weights Q, W, from the horizontal line AB, will be expressed by $l\,Q-Q\sqrt{a^2-d^2+2dx}+W\sqrt{a^2-d^2+2dx-x^2}$, divided by Q$+$W, which must be a maximum, in fluxions we have $-Q\times\frac{2d\dot{x}}{2\sqrt{a^2-d^2+2dx}}+\frac{2d\dot{x}-2x\dot{x}}{2\sqrt{a^2-d^2+2dx-x^2}}\times W=0$, whence $Q\times\frac{AB}{AC}=W\times\frac{EB}{EC}$, and $Q:W::\frac{EB}{EC}:\frac{AB}{AC}$.

COROL. 1. $Q:W::\sin\angle ECB:\sin\angle ACB$, for in the right angled triangle CEB, it will be $CB:1\text{ (radius)}::EB:\frac{EB}{CB}=\text{sine}\angle ECB$, whence the sine of the angle ABC is equal to $\frac{EC}{BC}$, therefore $AC:\frac{EC}{BC}::AB:\frac{EC\times AB}{AC\times BC}=\sin\angle ACB$, consequently $\frac{EB}{EC}:\frac{AB}{AC}::\frac{EB}{CB}:\frac{EC\times AB}{AC\times BC}$, that is $Q:W::\sin\angle ECB:\sin\angle ACB$.

COROL. 2. When $Q=w$, then the sine of ECB will be equal to the sine of ACB, and consequently the angle ACB, together with the angle ECB, make 180 degrees.

COROL. 3. If the weight w slides freely along the thread, then, as AC will be equal to BC, $EC=\frac{dw}{2\sqrt{4Q^2-w^2}}=$ (when $Q=w$) $\frac{d}{2\sqrt{3}}$.

COROL. 4. The weight Q will be the leaſt poſſible, when ACB is a right angle: For by Cor. 1. $\frac{w \times \text{ſin} \measuredangle ECB}{\text{ſin} \measuredangle ACB} = Q$, and will certainly be the leaſt when ſin $\measuredangle$ ACB is the greateſt poſſible, that is, when ACB is a right angle.

This problem may be anſwered by the reſolution of forces, thus:

Let the force, acting in the direction CB, be repreſented by F, which force may be divided into two others acting in the directions CE and BE, whence $\frac{EC \times F}{BC}$ and $\frac{EB \times F}{BC}$ are thoſe forces reſpectively. In the ſame manner may the force Q acting in the direction AC, be divided into the forces $\frac{EC \times Q}{AC}$, and $\frac{AE}{AC} \times Q$: Now it is very evident, that the ſum of the two forces acting in the direction CE, muſt be equal to w, and alſo that the forces acting in the oppoſite directions AE, EB muſt, in order that the weight C, may have no tendency to move horizontally, deſtroy each other; hence theſe equations, *viz.* $\frac{EC \times F}{BC} + \frac{EC \times Q}{AC} = w$, and $\frac{AE \times Q}{AC} = \frac{EB \times F}{BC}$, for BC in the firſt equation ſubſtitute its equal $\frac{AC \times EB \times F}{AE \times Q}$, and there will ariſe $\frac{EC \times Q}{AC} + \frac{AE \times EC \times Q}{AC \times EB} = w$, or (becauſe AE + BE = AB) $EC \times AB \times Q = AC \times EB \times w$, and conſequently $Q : w :: \frac{EB}{EC} : \frac{AB}{AC}$ the very ſame determination as before.

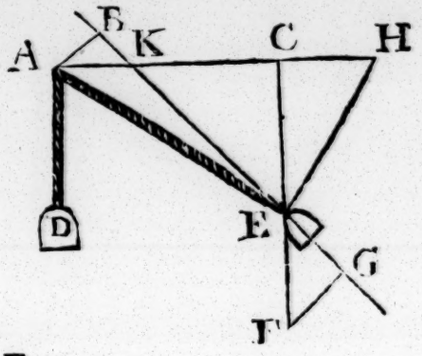
PROBLEM II.

IF at the ends of a thread DAE, moving upon the fixed tack A, there are hanged two weights D and E, whereof the weight E ſlides along the oblique plane BG, to find the place of the weight E where theſe weights are in equilibrio.

SOLUTION.

Let DAEH be the required poſition, and ſuppoſe the weight E, inſtead of being ſupported by the plane BG, to be ſuſtained by a thread EH, perpendicular to the ſaid plane, and faſtened to the point H; then

by the foregoing problem D : E : : ſin ∠ CEH : ſin ∠ AEH. Upon EK produced (if need be) let fall the perpendicular AB, which put $= a$, call the given part AK of the horizontal line AH, c, and BE, x. Then by plane trigonometry, $c : 1.$ (radius) $:: a : \frac{a}{c} =$ ſin ∠ AKB, and $\sqrt{a^2+x^2} : x :: 1 : \frac{x}{\sqrt{a^2+x^2}} =$ cos ∠ AEB equal to that of AEH, then $\frac{a}{c} : \frac{x}{\sqrt{a^2+x^2}} :: D : E$, whence $x = \frac{a^2 E}{\sqrt{D^2c^2-E^2a^2}}$.



Otherwiſe,

Let l be put for the length of the ſtring EAD, the other letters remaining as before, we ſhall have $AE = \sqrt{a^2+x^2}$, $AD = l - \sqrt{a^2+x^2}$, and $KE = x - \sqrt{c^2-a^2}$; hence by Lemma 1, $lD - D\sqrt{x^2+a^2} + aEx - \frac{\sqrt{c^2-a^2}\,aE}{c}$, expreſſes the perpendicular diſtance of the common center of gravity of the weights D, E from the horizontal line AH, which (becauſe a maximum) being put into fluxions and properly reduced, gives $x = \frac{a^2 E}{\sqrt{D^2c^2-E^2a^2}}$, the ſame as before.

Sir *Iſaac Newton*'s method of ſolution, ſee his Univerſal Arithmetic, *p.* 157.

Suppoſe the problem done, and parallel to AD, draw EF, which let be to AE as the weight E to the weight D. And from the points A and F, to the line BG, let fall the perpendiculars AB, FG. Now ſince the weights are by ſuppoſition, as the lines AE and EF, expreſs thoſe weights by thoſe lines, the weight D by the line EA, and the weight E by the line EF. Therefore the body E, directed by the force of its own weight EF, tends towards F, and by the oblique force EG tends towards G; and the ſame body E, by the direct force AE of the weight D, is drawn towards A, and by the oblique force BE is drawn towards B. Since therefore the weights ſuſtain each other in equilibrio, the force by which the weight E is drawn towards B, ought to be equal to the contrary force by which it tends towards G, that is, BE ought to be equal to EG. But now the ratio of AE to EF is given by the hypotheſis; and by reaſon of the given angle FEG, there is alſo given the ratio of FE to EG, to which

BE is equal. Therefore there is given the ratio of AE to BE; AB is alſo given in length, and thence the triangle ABE, and the point E will be eaſily given, *viz.* make $AB = a$, $BE = x$, and AE will be equal to $\sqrt{a^2 + x^2}$, moreover let AE be to BE in the given ratio of d to e, and $e\sqrt{a^2+x^2}$ will be $= dx$, and the parts of the equation being ſquared and reduced, $eeaa = ddxx - eexx$, or $x = \frac{ae}{\sqrt{dd-ee}}$. Therefore the length BE is found, which determines the place of the weight E.

By the foregoing ſolution it appears, that $x = \frac{a^2E}{\sqrt{D^2c^2-E^2a^2}}$, but in order to ſhew that theſe values of x are exactly the ſame, we muſt ſubſtitute for D, E, &c. their reſpective values in terms of d and e. Now AE : BE :: d : e, and EF : AE :: E : D by the hypotheſis, but FE : EG :: AK : AB, by the ſimilar triangles EFG, ABK, wherefore AE : EG :: $\frac{D}{E} : \frac{AB}{AK}$, or as d : e, becauſe BE = EG, conſequently $\frac{D}{E} : \frac{a}{e}$ :: d : e, whence $\frac{D^2}{E^2} : \frac{a^2}{c^2}$ :: $d^2 : e^2$, and again $\frac{D^2}{E^2} - \frac{a^2}{c^2} : \frac{a^2}{c^2}$ :: $d^2 - e^2 : e^2$,

$\therefore \frac{e^2}{d^2-e^2} = \frac{a^2E^2}{D^2c^2-a^2E^2}$ and $\frac{e}{\sqrt{d^2-e^2}} = \frac{aE}{\sqrt{D^2c^2-a^2E^2}}$, therefore

$\frac{ea}{\sqrt{d^2-e^2}} = \frac{a^2E^2}{\sqrt{D^2c^2-E^2a^2}}$.

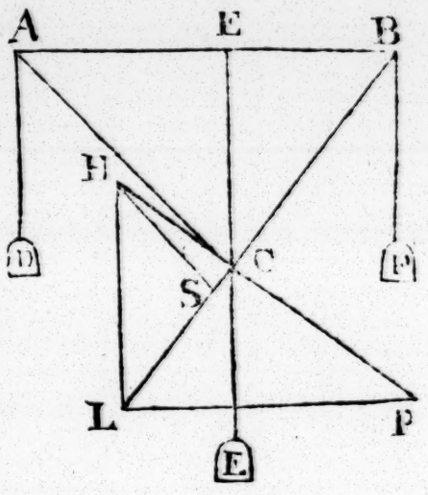
PROBLEM III.

IF on the ſtring DACBF, that ſlides about the two tacks A, B, placed in the horizontal line AB, there are hung three weights D, E, F. D and F, at the ends of the ſtring, and E at its middle point C, placed between the tacks: From the given weights and diſtance AB, to find the ſituation of the point C, where the middle weight hangs, when the three weights are in equilibrio.

SOLUTION.

From C draw CE perpendicular to AB. Put AB = d, the ſtring CBF = l, EC = y, BE = x; then AE = $d-x$, and AC = $\sqrt{d-x^2+y^2}$, whence

whence $AD = L - \sqrt{d - x^2 + y^2}$ (L being put for the ſtring CAD) and $BC = \sqrt{x^2 + y^2}$, conſequently $BF = l - \sqrt{x^2 + y^2}$, and (by Lemma 1.) the perpendicular diſtance of the common center of gravity of the weights D, E, F, from the horizontal line AB, will be expreſſed by $\frac{L \times D - D \times \sqrt{d - x^2 + y^2} + y \times E + l \times F - F \times \sqrt{x^2 + y^2}}{D + E + F}$, which muſt be a maximum; in fluxions with x conſtant, gives $\frac{-Dy\dot{y}}{\sqrt{d - x^2 + y^2}} + E\dot{y} - \frac{Fy\dot{y}}{\sqrt{x^2 + y^2}} = 0$, whence $E - \frac{D \times EC}{AC} = \frac{F \times EC}{BC}$, and by taking the fluxion of the ſame expreſſion with y invariable, we have $\frac{-D \times \dot{d - x} \times -\dot{x}}{\sqrt{d - x^2 + y^2}} + \frac{-F \times \dot{x}}{\sqrt{x^2 + y^2}} = 0$, or $\frac{D \times AE}{AC} = \frac{F \times BE}{BC}$.



COROL. 1. $F : E :: AE \times BC : AB \times EC :: \text{ſin} \measuredangle ACE : \text{ſin} \measuredangle ACB$, and $D : E :: \text{ſin} \measuredangle ECB : \text{ſin} \measuredangle ACB$.

COROL. 2. If ACB be a right angle, then $D : F :: EC : AC$.

COROL. 3. Through C let HCP be drawn perpendicular to BC, and from any point, as L in BC produced, draw LH and LP, the former perpendicular, and the latter parallel to the horizon; then if a heavy body be ſuſtained upon the inclined plane HCP, by a power acting in any given direction AC, to which HS is drawn parallel, the ſuſtaining power, weight of the body, and its preſſure upon that plane, will be reſpectively as SH, HL and LS: For if the weight E repreſents a heavy body, ſupported upon the inclined plane HCP, by means of the weight D acting in the direction AC, its preſſure upon that plane will be expounded by the weight F, acting in the direction CB; this being premiſed, it appears by Corol. 1, that $F : E :: \text{ſin} \measuredangle ACE : \text{ſin} \measuredangle ACB$, and $D : E :: \text{ſin} \measuredangle ECB : \text{ſin} \measuredangle ACB$, whence D; E and F are reſpectively as the ſines of the angles ECB, ACB and ACE. But the angle ECB is equal to the angle HLS, the angles ACB, HSC are equal, and the angle ACE is equal to the angle LHS; therefore the ſines of the angles of the triangle HLS,

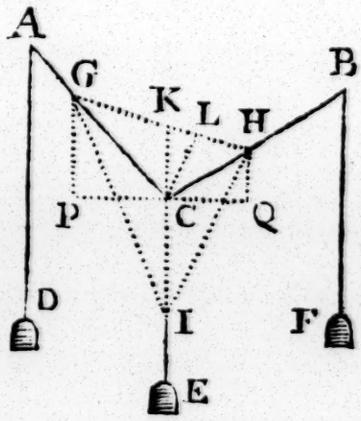
and conſequently the ſides HS, HL and LS, are as the weights D, E and F.

Corol. 4. If a heavy body be ſuſtained upon an inclined plane, by a power acting in a direction parallel to that plane, then the weight of the body, the power that ſuſtains it, and its preſſure againſt the plane, are reſpectively as the length HP, height LH, and baſe LP of that plane. For in this circumſtance HS coincides with HC, and LS with LC, therefore D, E and F, are as HC, HL and LC; but $HC \times HP = HL^2$ and $HP \times LC = HL \times LP$ by the ſimilar triangles HLP, HLS and LCP, therefore E, D and F are as HP, HL and LP.

Corol. 5. If a heavy be ſuſtained upon an inclined plane by a power acting parallel to the horizon, then the preſſure on the plane, the power and the weight, are reſpectively as radius, the ſine and coſine of the plane's elevation. For in this caſe HS is parallel to LP, and therefore the triangle LHS now becomes ſimilar to HLP, and conſequently LS, HS and HL, are as HP, HL and LP, that is, as radius, the ſine of the angle HPL, and ſine of the angle LHP.

Corol. 6. The power D is leaſt when its line of direction is parallel to the horizon.

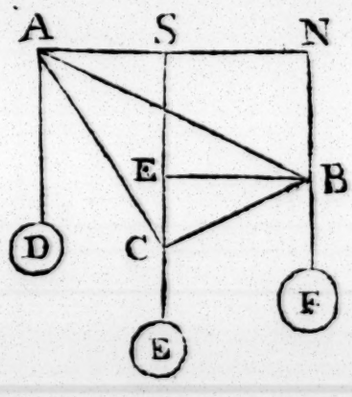
Sir *Iſaac Newton*'s method of ſolution to this problem, ſee his Univerſal Arithmetic, *p.* 159.



Since the tenſion of the thread AC is equal to the tenſion of the thread AD, and the tenſion of the thread BC to the tenſion of the thread BF, the tenſions of the ſtrings or threads AC, BC, EC, will be as the weights D, E, F. Then take the parts of the thread CG, CH, CI, in the ratio as the weights, compleat the triangle GHI. Produce IC till it meet GH in K, and GK will be equal to KH, and $CK = \frac{1}{2}CI$, and conſequently C the center of gravity of the triangle GHI. For draw PQ through C, perpendicular to CE, and perpendicular to that, from the points G and H, draw GP, HQ. And if the force by which the thread AC, by the force of the weight D, draws the point C towards A, be expreſſed by the line GC, the force by which that thread will draw the ſame point towards P, will be expreſſed by the line CP, and the force by which it draws towards K, will be expreſſed by the line GP. And in like manner, the forces by which the thread BC, by means of the weight F, draws the ſame point C towards B, Q and K, will be expreſſed by the lines CH, CQ and HQ; and the force by which the thread CE, by

means of the weight E, draws that point C towards E, will be expreſſed by the line CI. Now ſince the point C is ſuſtained in equilibrio by equal forces, the ſum of the forces by which the threads AC and BC do together draw C towards K, will be equal to the contrary force by which the thread EC draws that point towards E; that is, the ſum of GP + HQ will be equal to CI; and the force by which the thread AC draws the point C towards P, will be equal to the contrary force by which the thread BC draws the ſame point C towards Q; that is, the line PC is equal to the line CQ. Wherefore ſince PG, CK and QH are parallel, GK will alſo = KH, and $CK \left(= \frac{GP+HQ}{2}\right) = \frac{1}{2}CI$, which was to be ſhewn. It remains therefore to determine the triangle GCH, whoſe ſides GC and HC are given, together with the line CK, which is drawn from the vertex C to the middle of the baſe. Let fall therefore from the vertex C to the baſe CH the perpendicular CL, and $\frac{\overline{GC}^2 - CH^2}{2GH}$ will be $= KL = \frac{GC^2 - KC^2 - GK^2}{2GK}$. For 2 GK write GH, and having rejected the common diviſor GH, and ordered the terms, you will have $CG^2 - 2KC^2 + CH^2 = 2GK^2$ or $\sqrt{\frac{1}{2}\overline{GC}^2 - KC^2 + \frac{1}{2}CH^2} = GK$. Having found GK, or KH, there are given together the angles GCK, KCH, or DAC, FBC. Wherefore, from the points A and B, in theſe given angles DAC, FBC, draw the lines AC, BC, meeting in the point C; and C will be the point ſought.

When the points A and B are in the ſame horizontal line, this ſolution agrees exactly with the former. But if a general ſolution be required by the firſt method, when the points A and B have any given ſituation out of the horizontal poſition, then from the point A draw ASN parallel to the horizon, and from the point B draw BN perpendicular to AN; draw alſo CS parallel to FBN, and BE parallel to AN. Put $AB = d$, $AN = a$, $BN = b$, L and l for the reſpective lengths of the ſtrings CAD, CBF, $CE = y$, $BE = x$. Hence we get $CB^2 = \sqrt{x^2 + y^2}$, $AC^2 = \sqrt{\overline{b+y}^2 + \overline{a-x}^2}$ and $AD = L - \sqrt{\overline{b+y}^2 + \overline{a-x}^2}$, alſo $NF = b + l - \sqrt{x^2 + y^2}$, and $CS = b + y$; conſequently, by Lemma 1, the perpendicular diſtance of the common center of gravity of the three weights D, E, F, from the horizontal line ASN, will be expreſſed by



$$\frac{E\times D-D\times\sqrt{b+y^2+a-x^2}+F\times b+F\times l-F\times\sqrt{x^2+y^2}+E\times b+E\times y}{D+E+F},$$

which, becauſe a maximum, being put into fluxions, with y conſtant, and the equation thence ariſing properly ordered, gives $D\times\frac{AS}{AC}=F\times\frac{SN}{CB}$; and the expreſſion for the maximum being again put into fluxions, with x conſtant, will, when properly reduced, give $D\times\frac{CS}{AC}+F\times\frac{CE}{CB}=E$. But theſe equations may be otherwiſe derived by the reſolution of forces, thus: Let the force D, acting in the direction AC, be divided into two other forces acting reſpectively in the directions AS, CS, thoſe forces will be denoted by $D\times\frac{AS}{AC}$ and $D\times\frac{CS}{AC}$; in the ſame manner by reſolving the force F, acting in the direction BC, into the forces acting in the directions CE, BE, we ſhall have $F\times\frac{BE}{CB}$ and $F\times\frac{CE}{CB}$ for thoſe forces reſpectively. Now it is very evident, the forces acting in the contrary and parallel directions AS, BE, muſt deſtroy each other, otherwiſe the weights would have a tendency to move horizontally, therefore $D\times\frac{AS}{AC}=F\times\frac{SN}{CB}$. And again, it is certain that the ſum of thoſe parts of the forces D and F, which act in the direction CS, muſt be equal to the abſolute force of gravity or weight E; hence this equation $D\times\frac{CS}{AC}+F\times\frac{CE}{CB}=E$, which, together with the other before found, are exactly the ſame as thoſe above determined, and may be eaſily ſhewn to agree with the foregoing method of ſolution given by Sir *Iſaac Newton*.

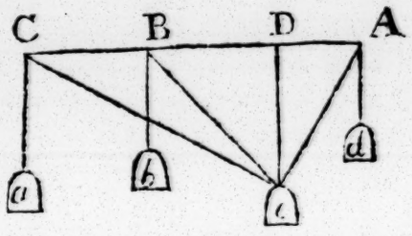
PROBLEM IV.

LET three weights *a*, *b*, *d*, ſupport the weight C, after the manner deſcribed in the laſt problem; to determine their poſition when in equilibrio.

SOLUTION.

Suppoſe it done, and C *c* A the poſition required. From *c* draw *c* D perpendicular to the horizontal line AC, put AC = *q*, CB = *n*, *l*, *l*, and

and l'' for the reſpective lengths of the ſtrings cBb, cCa, and cAd, $BD=x$, $cD=y$. Then will $Bb=l-\sqrt{x^2+y^2}$, $Bc=\sqrt{x^2+y^2}$, $cC=\sqrt{n+x^2+y^2}$, $cA=\sqrt{m-x^2+y^2}$ $(m=q-n)$, $Ca=l'-\sqrt{n+x^2+y^2}$, and $Ad=l''-\sqrt{m-x^2+y^2}$. By Lemma 1, the perpendicular diſtance of the common center of gravity of the four weights a, b, c, d, from the horizontal line AC, will be expreſſed



by $\frac{bl-b\sqrt{x^2+y^2}+al'-a\sqrt{n+x^2+y^2}+dl''-d\sqrt{m-x^2+y^2}+cy}{a+b+c+d}$,

the fluxion of which, making y conſtant, is $\frac{bx\dot{x}}{\sqrt{x^2+y^2}}+\frac{a\overline{n+x}\,\dot{x}}{\sqrt{n+x^2+y^2}}-\frac{d\overline{m-x}\,\dot{x}}{\sqrt{m-x^2+y^2}}=0$. And by taking the fluxion of the ſame expreſſion, with x conſtant, gives $\frac{by\dot{y}}{\sqrt{x^2+y^2}}+\frac{ay\dot{y}}{\sqrt{n+x^2+y^2}}+\frac{dy\dot{y}}{\sqrt{m-x^2+y^2}}-c\dot{y}=0$, from theſe equations, properly reduced, we have $b\times\frac{BD}{Bc}+a\times\frac{CD}{Cc}=d\times\frac{DA}{Ac}$ and $b\times\frac{De}{Bc}+a\times\frac{Dc}{Cc}+d\times\frac{Dc}{Ac}=c$.

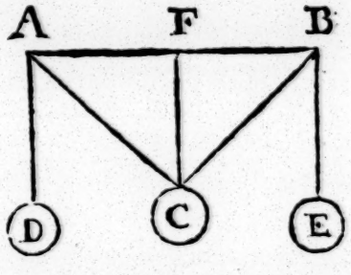
The ſame concluſions may be otherwiſe derived by the application of the method of dividing forces, thus: The force a may be divided into two other forces, acting in the directions CD, cD, the force b may be divided into the forces acting in the directions BD, cD, and the force d into thoſe acting in the directions AD, cD, whence we have $a\times\frac{CD}{Cc}+b\times\frac{DB}{Bc}=d\times\frac{DA}{Ac}$, becauſe the forces acting on each ſide the point D, in the direction AC, muſt be equal; otherwiſe the weight c would have a tendency to move horizontally. Again, $a\times\frac{Dc}{Cc}+b\times\frac{Dc}{Bc}+d\times\frac{Dc}{Ac}$ muſt be equal to c, becauſe thoſe parts of the abſolute forces a, b, d, which act in the common direction cD, muſt together be equal to the whole weight c, which they ſupport.

PROBLEM V.

TO the ends D and E of the ſtring DACBE (paſſing over two fixed tacks at the points A, B, in the horizontal line AB) let the weights D and E be appended; 'tis required to find the poſition of the ſaid ſtring, when a third weight C, (ſliding freely on the ſtring between the tacks) ſhall be a juſt counterpoiſe to the former two.

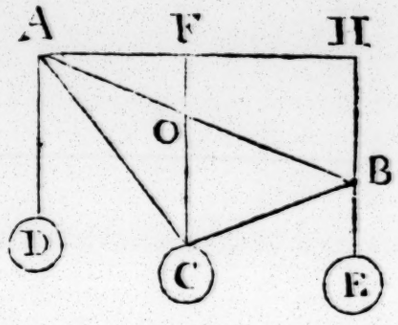
SOLUTION.

Let ACB be the required poſition; from C draw CF perpendicular to AB. Put AB $= n$, l for the length of the ſtring DACBE, AF $= x$, AC $= y$, AD $= z$. Then $\sqrt{x^2+y^2} =$ FC, $n-x =$ FB, $\sqrt{n^2+y^2-2nx} =$ CB, and $l-y-\sqrt{n^2+y^2-2nx}-z =$ BE; conſequently $\frac{D\times z+\sqrt{y^2-x^2}\times C+\overline{l-y}\times E-}{D+C+E}$



$\frac{z\times E-E\times\sqrt{y^2+n^2-2nx}}{D+C+E}$ expreſſes the perpendicular diſtance of the common center of gravity of the three weights D, C, E, from the horizontal line AB, which muſt be a maximum; the fluxion of this expreſſion, with x and y conſtant, gives $D\dot{z}-E\dot{z}=0$; Whence D $=$ E, the ſame expreſſion in fluxions with x and z conſtant, gives $\frac{C\times y\dot{y}}{\sqrt{y^2-x^2}}-E\times\dot{y}-\frac{E\times y\dot{y}}{\sqrt{y^2+n^2-2nx}}=0$; or, when reduced, $\frac{AC}{FC}\times C=\frac{AC+CB}{CB}\times E$, and in fluxions with x only variable, we have $\frac{C\times\dot{x}}{\sqrt{y^2-x^2}}+\frac{nE\dot{x}}{\sqrt{n^2+y^2-2nx}}=0$, whence $\frac{AF}{FC}\times C=\frac{AB}{CB}\times E$, conſequently $\frac{AC+CB}{AC}=\frac{AB}{AF}$, which can only happen when AC $=$ CB, and AF $=$ FB. Now becauſe D $=$ E, it follows, that if the middle weight C be ſuffered to ſlide freely along the ſtring, no equilibrium can enſue, unleſs the ſuſtaining weights D and E are equal, for otherwiſe the greater weight will deſcend continually until the leſſer be quite pulled up to the horizontal line AB. Either of the foregoing equations expreſſing the relation of C to E, will, by ſubſtituting AC for CB, and AF for FB, become $C\times CB=2FC\times E$,

and consequently C : E : : 2 FC : CB. If it were required to determine the position of the string when the weights D, C, E, are in equilibrio, supposing one of the points as B situated out of the horizontal line AFH; then having drawn BH perpendicular to AH, and CF parallel to BH, join the points F, B, with the right line FB; put AH = m, HB = r, AB = n, FA = x, AC = y, AD = z, l for the length of the string DACBE, then FC = $\sqrt{y^2-x^2}$, FH = $m-x$, FB = $\sqrt{m-x^2+r^2}$, and by plane trigonometry, $\sqrt{m-x^2+r^2}$: 1 (radius) : : r : $\frac{r}{\sqrt{m-x^2+r^2}}$ = sin $<$BFH = cos $<$CFB, whence (by Lemma 2.) we have $y^2-x^2+m-x^2+r^2-2\sqrt{y^2-x^2}\times\sqrt{m-x^2+r^2}\times\frac{r}{\sqrt{m-x^2+r^2}}$ = CB^2, which contracted becomes $\sqrt{y^2+m^2-2mx+r^2-2r\sqrt{y^2-x^2}}$ = CB. Hence $\frac{Dz+\sqrt{y^2-x^2}\times C+\overline{l-z}\times E-y\times E-E\times\sqrt{y^2+m^2-2mx+r^2-2r\sqrt{y^2-x^2}}}{D+C+E}$ = to the perpendicular distance of the common center of gravity of the three weights from the horizontal line AH (by Lemma 1.) which must be a maximum in fluxions with x and y constant, gives $D\dot{z}-E\dot{z}=0$, whence D = E. Now let the same expression be put into fluxions with z and x constant, and there will arise $C\times\frac{y\dot{y}}{\sqrt{y^2-x^2}}-E\dot{y}=\frac{\sqrt{y^2-x^2}\times y\dot{y}}{\sqrt{y^2-x^2}\times\sqrt{y^2+m^2-2mx+r^2-2r\sqrt{y^2-x^2}}}\times E$ or $C\times\frac{AC}{FC}=\frac{AC\times FC-HB\times AC+FC\times CB}{FC\times CB}\times E$; if x only flows, we shall have $\frac{-x\dot{x}}{\sqrt{y^2-x^2}}\times C=\frac{-m\dot{x}\sqrt{y^2-x^2}+rx\dot{x}\times E}{\sqrt{y^2-x^2}\times\sqrt{y^2+m^2-2mx+r^2-2r\sqrt{y^2-x^2}}}$, that is $\frac{-AF}{FC}\times C=\frac{HB\times AF-AH\times FC}{FC\times CB}\times E$, consequently $\frac{FC\times AH-HB\times AF}{AF}=\frac{AC\times FC-HB\times AC+FC\times CB}{AC}$, which properly reduced becomes $\frac{FH}{CB}=\frac{AF}{AC}$. It now remains to shew, that when this equation obtains,



the weight C will be in the loweſt point poſſible; that is, the line CF, perpendicular to AH, will be a maximum, let the ſituation of D and E be what they will. Put AO $= x$, OC $= y$, c for the coſine of the angle COB to the radius, 1, then that of the angle COA will be denoted by $-c$. By Lemma 2. we have AC equal to $\sqrt{x^2+y^2+2cxy}$, BC $= \sqrt{\overline{n-x}^2+y^2-2cy\overline{n-x}}$, conſequently their ſum will be equal to ACB. By the ſimilar triangles ABH, AOF, we have AB : BH :: AO : OF $= \frac{rx}{n}$, hence CF $= y + \frac{rx}{n}$, which muſt be a maximum, therefore $\frac{r\dot{x}}{n} = -\dot{y}$. Now by taking the fluxion of ACB, we have $\frac{x\dot{x}+v\dot{y}+cy\dot{x}+cx\dot{y}}{\sqrt{x^2+y^2+2cxy}} + \frac{y\dot{y}-\overline{n-x}\dot{x}-\overline{n-x}c\dot{y}+cy\dot{x}}{\sqrt{\overline{n-x}^2+y^2-2cy\overline{n-x}}} = 0$; in this equation, for $\dot{y}$ put its value, *viz.* $\frac{-r\dot{x}}{n}$ or $-c\dot{x}$ (becauſe $\frac{r}{n} = c$) we get $\frac{x-c^2x}{AC} = \frac{n-x-\overline{n-x}cc}{BC}$; but becauſe $1-cc = \text{ſin} < AOF$, it follows, that $x-cc\,x =$ AF, and $n-x-\overline{n-x}cc$ or $\overline{1-cc} \times \overline{n-x} =$ FH, therefore $\frac{AE}{AC} = \frac{FH}{BC}$, the very ſame equation as was before determined.

Corol. 1. If the point B coincides with the point H, then (CH being drawn) $\frac{AF}{AC} = \frac{FH}{CH}$, or AF : FH :: AC : HC, which can only be true when ACH is an iſoſceles triangle, or AF $=$ FH, and AC $=$ HC.

Corol. 2. Becauſe AF : FH :: AC : BC, by the above inveſtigation, and AF : FH :: AO : BO, by ſimilar triangles, it follows that AO : BO :: AC : BC, and therefore the angle ACB is biſected by CF drawn perpendicular to the horizontal line AH. Hence the truth of what was advanced at Corol. 3. Prob. I. becomes manifeſt, *viz.* That if the weight w, (ſee *fig.* to Prob. I.) ſlides freely along the thread, then AC $=$ CB. It alſo appears, that the three weights D, C, E, being ſuſtained in equilibrio, by means of a ſtring DACBE, along which the middle weight freely ſlides, not only the two weights D, E, muſt be equal, but alſo that the middle one C will be in the loweſt point poſſible between A and B, and remain conſtantly in that point independent of the ſituation of the equal weights D, E. But if we ſuppoſe the middle weight C, faſtened to a certain point in that part of the ſtring upon which it was before ſuffered to ſlide, then if the weights D, E, are equal, thoſe weights and the middle

one

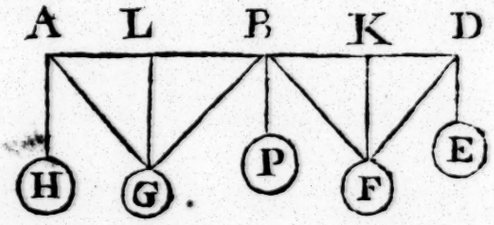
one C, when in equilibrio, will obtain the very ſame ſituation as in the former caſe; but otherwiſe, the middle weight will be higher than the loweſt point, and, by its endeavour to deſcend, will ſo act upon the greater weight, as to become a juſt counterpoiſe to both the weights D, E, acting in their proper directions.

PROBLEM VI.

TO determine the poſition of the ſtring HAGBFDE, paſſing over three ſmall pullies fixed in the points A, B, D of the horizontal line ABD, when the given weights H, E, appended to the ends thereof, ſuſtain two other known weights G, F, (taſtened to the ſtring at the points G, F) in equilibrio.

SOLUTION.

From the points G, F, draw GL and FK, each perpendicular to ABD; put $AB = d$, $BD = b$, the ſtring $GAH = a$, $FDE = b$ and $GBF = n$, $BF = x$, ſin $< ABG = z$, ſin $< DBF = y$, radius $= 1$; then, by Lemma 2, $AG = \sqrt{d^2 + \overline{n-x}^2 - 2dz \times \overline{n-x}}$, whence $AH = a - \sqrt{d^2 + \overline{n-x}^2 - 2dz \times \overline{n-x}}$, $LG = \overline{n-x} \times \sqrt{1-z^2}$ and $DE = b - \sqrt{b^2 + x^2 - 2bxy}$, $KF = x \times \sqrt{1-y^2}$; therefore, by Lemma 1, the perpendicular diſtance of the common center of gravity of all the weights from the line AD is $= \frac{aH - H \times \sqrt{d^2 + \overline{n-x}^2 - 2dz \times \overline{n-x}} + G \times \overline{n-x}\sqrt{1-z^2}}{H+G+F+E}$



$\frac{+ F \times \sqrt{1-y^2} + bE - E\sqrt{b^2+x^2-2bxy}}{H+G+F+E}$, which being put into fluxions with x and y conſtant, and properly reduced, gives $H \times \frac{AB}{AG} = G \times \frac{LB}{LG}$: By taking the fluxion of the ſame expreſſion with x and z conſtant, we get $F \times \frac{xy\dot{y}}{\sqrt{1-y^2}} = E \times \frac{bx\dot{y}}{\sqrt{b^2+x^2-2bxy}}$, that is $F \times \frac{BK}{KF}$

$= E \times \frac{BD}{FD}$. Now let only x flow, then we have $-H \times \frac{2\overline{n-x} \times -\dot{x} - 2dz \times -\dot{x}}{2\,AG} + G \times \sqrt{1-x^2} \times -\dot{x} + F \times \sqrt{1-y^2} \times \dot{x} - E \times \frac{2x\dot{x} - 2by\dot{x}}{2\,FD} = 0$; divide the whole by $\dot{x}$, for z and y ſubſtitute their reſpective values $\frac{LB}{GB}$ and $\frac{BK}{FB}$, we then have $H \times \frac{BG}{AG} - H \times \frac{AB \times LB}{AG \times BG} + F \times \frac{FK}{BF} + E \times \frac{BD \times BK}{BF \times FD} - E \times \frac{BF}{FD} = G \times \frac{GL}{BG}$. But in order to contract this equation as much as may be, let the firſt of the foregoing equations, *viz.* $H \times \frac{AB}{AG} = G \times \frac{LB}{LG}$ be multiplied by $\frac{GL^2}{BG \times AG \times LB}$, we have $H \times \frac{AB \times LG^2}{AG \times LB \times BG} = G \times \frac{GL}{BG}$; and by multiplying the ſecond, *viz.* $F \times \frac{BK}{KF} = E \times \frac{BD}{FD}$, by $\frac{FK^2}{BK \times BF}$ we get $F \times \frac{FK}{BF} = E \times \frac{BD \times FK^2}{FD \times BK \times BF}$; for $F \times \frac{FK}{BF}$ and $G \times \frac{GL}{BG}$ in the above equation, ſubſtitute their equals thus found, and we ſhall have $H \times \frac{BG}{AG} - H \times \frac{AB \times BL}{AG \times BG} - H \times \frac{AB \times LG^2}{AG \times LB \times BG} = E \times \frac{BF}{FD} - E \times \frac{BD \times BK}{BF \times FD} - E \times \frac{BD \times FK^2}{FD \times BK \times BF}$, which properly reduced becomes $H \times \frac{AL \times BG}{AG \times LB} = E \times \frac{DK \times BF}{FD \times BK}$. The ſame concluſions may be eaſily obtained by help of the 3d Problem; for if P be the weight which, being appended to the ſtring CBP, would ſuſtain the weights H and G in equilibrio, it is very certain that the ſame weight P, being appended to the ſtring FBP, would alſo be a juſt counterpoiſe to the other two F, E; and conſequently, by the above cited Problem, $\frac{H \times AL}{AG} = P \times \frac{BL}{BG}$ and $P \times \frac{BK}{BP} = E \times \frac{DK}{DF}$; exterminate P, we have $H \times \frac{AL \times BG}{BL \times AG} = E \times \frac{DK \times BF}{FD \times BK}$. Again by Corol. 1. of the ſame Problem, $E : F :: BK \times FD : BD \times FK$, alſo $H : G :: LB \times AG : AB \times GL$, whence $H \times \frac{AB}{AG} = G \times \frac{LB}{LG}$, and $F \times \frac{BK}{FK} = E \times \frac{BD}{FD}$, the very ſame equations as before.

COROL. 1. If AGB be a right angle, and E = o, then, by the nature of the problem, F, by its abſolute weight, muſt ſuſtain G and H in equilibrio, and conſequently F : G :: GL : BG.

COROL. 2. If AGB and BED are right angles, the above general equations will become $H \times \frac{AB}{AG} = G \times \frac{LB}{LG}$, $F \times \frac{BK}{KF} = E \times \frac{BD}{FD}$, and $F \times \frac{KF}{BF} = G \times \frac{GL}{BG}$, whence $H : E :: \frac{KD}{FK} : \frac{AL}{LG}$ :: tan $\sphericalangle$ FBK : tan $\sphericalangle$ GBL.

COROL. 3. If two bodies G, F, ſuſtain each other upon the inclined planes GF, FB, by means of the ſtring GBF, whoſe parts GB, FB, are parallel to thoſe planes reſpectively, then the preſſures of the bodies G, F, thereon, will be in the direct ratio of the cotangents of the angles of elevation of the planes above the horizon. For H and E repreſent the preſſures of the bodies G and F againſt the propoſed planes; and theſe are by the laſt Corol. in the ratio of the tangents of the angles FBK, GBL; therefore, as the tangents of angles are reciprocally as their cotangents, it follows, that H is to E in the direct ratio of the cotangents of elevation.

COROL. 4. The general relation of H to E, the angles AGB, BFD, being either obtuſe or acute, is that of $\frac{\text{ſin } \sphericalangle AGL}{\text{ſin } \sphericalangle BGL}$ to $\frac{\text{ſin } \sphericalangle DFK}{\text{ſin } \sphericalangle BFK}$. For in the general equation $H \times \frac{AL \times BG}{AG \times LB} = E \times \frac{DK \times BF}{FD \times BK}$. $\frac{AL}{AG}$ and $\frac{BL}{BG}$ are reſpectively the ſines of the angles AGL, BGL, as are alſo $\frac{BK}{BF}$, $\frac{DK}{DF}$ thoſe of the angles BFK and DFK, radius being unity.

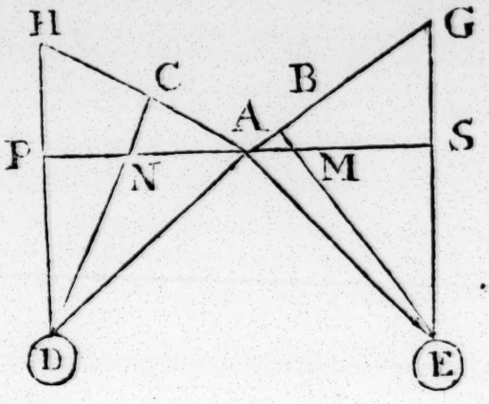
PROBLEM VII.

IF at the ends of the thread DAE, moving upon the fixed tack A, there are hanged two weights D and E, which deſcend through the oblique lines CD, BD, to find where theſe weights are in equilibrio upon thoſe oblique lines.

SOLUTION.

Suppoſe the problem done, and DAE the poſition required. From the point A to the oblique lines, let fall the perpendiculars AC, AB, and let

the lines EG, DH, erected from the weights perpendicularly to the horizon, meet them in the points G and H; through the point A draw SP parallel to the horizon, meeting the oblique lines DC, EB, in the points N and M. Put c for the length of the ſtring DAE, a=AB, b=AC, d=BM, let BE : EG :: 1 : r, and CD : DH :: 1 : s, CN = b, AE = x, then will AD = $c-x$, BE = $\sqrt{x^2-a^2}$, ME = $\sqrt{x^2-a^2}-d$. But ME : ES :: EG : BE, by the ſimilar triangles BEG, MES, that is, ES = $\frac{\sqrt{x^2-a^2}-d}{r}$, and by the ſame way of reaſoning we have DP = $\frac{\sqrt{c-x^2-b^2}-b}{s}$; therefore DP × D + SE × E, the whole divided by D + E, which, by Lemma 1, is the diſtance of the common center of gravity of the weights, from the horizontal line PAS, will be expreſſed by $\frac{D\times\sqrt{c-x^2-b^2}-D\times b}{s\times\overline{D+E}}+\frac{E\times\sqrt{x^2-a^2}-E\times d}{r\times\overline{D+E}}$, the fluxion of this being made equal to nothing, and the equation properly ordered, gives $D\times\frac{c-x}{s\sqrt{c-x^2-b^2}}=E\times\frac{x}{r\sqrt{x^2-a^2}}$, or $D\times\frac{AD}{s\times CD}=E\times\frac{AE}{r\times BE}$.



Sir *Iſaac Newton*'s method of ſolution to this problem, ſee *page* 158 of his Univerſal Arithmetic.

The force by which the weight E endeavours to deſcend in a perpendicular line; that is, the whole gravity of E will be to the force by which the ſame weight endeavours to deſcend in the oblique line BE, as GE to BE; and the force by which it endeavours to deſcend in the oblique BE, will be to the force by which it endeavours to deſcend in the line AE, that is, to the force by which the thread AE is diſtended or ſtretched, as BE to AE; and conſequently the gravity of E will be to the tenſion of the thread AE, as GE to AE; and by the ſame reaſon the gravity of D will be to the tenſion of the thread AD, as HD to AD. Let therefore the length of the whole DA + AE be c, and let its part AE be = x, and its other part AD will be = $c-x$. And becauſe $AE^2-AB^2=BE^2$, and $AD^2-AC^2=CD^2$; let moreover AB = a, and AC = b, and BE will be = $\sqrt{x^2-aa}$, and CD = $\sqrt{x^2-2cx+cc-bb}$. Farther, ſince the triangles BEG, CDH, are

are given in ſpecie, let BE : EG : : f : E, and CD : DH : : f : g, and EG will be equal to $\frac{E}{f} \times \sqrt{x^2 - a^2}$, and $DH = \frac{g}{f} \times \sqrt{xx - 2cx + cc - bb}$. Wherefore ſince GE : AE : : weight E : tenſion of AE, and HD : AD : : weight D : tenſion of AD, and thoſe tenſions are equal, you will have

$$\frac{Ex}{\frac{E}{f} \times \sqrt{x^2 - a^2}} = \text{tenſion of AE} = \text{tenſion of AD} = \frac{Dc - Dx}{\frac{g}{f} \times \sqrt{x^2 - 2cx + cc - bb}},$$

from the reduction of which equation there comes out $g\,x\,x \times$ into

$$\sqrt{xx - 2cx + cc - bb} = \overline{Dc - Dx} \times \sqrt{xx - aa}, \text{ or } \begin{matrix} \frac{gg}{-DD} & x^4 & \begin{matrix} -2gge \\ +2DDc \end{matrix} & x^3 & \begin{matrix} +gg\,cc \\ -gg\,bb \\ -DD\,cc \\ +DDaa \end{matrix} & x^2 \end{matrix}$$

$$- 2DD\,c\,aa\,x + DD\,cc\,aa = 0.$$

If you deſire a caſe wherein the problem may be conſtructed by a rule and compaſs, make the weight D to the weight E, as the ratio $\frac{BE}{EG}$ to the ratio $\frac{CD}{DH}$, and g will become = D; and ſo in the room of the precedent equation, you will have this $\frac{aa}{-bb}\,xx - 2\,a\,cx + aa\,cc = 0$, or $x = \frac{ac}{a+b}$.

The above equation, *viz.* $\frac{Ex}{\frac{E}{f} \times \sqrt{x^2 - a^2}} = \frac{Dc - Dx}{\frac{g}{f} \times \sqrt{x^2 - 2cx + cc - bb}}$, or $\frac{E \times AE}{\frac{E}{f} \times BE} = \frac{D \times AD}{\frac{g}{f} \times DC}$, may be eaſily ſhewn to agree with that derived by our method, by writing r for $\frac{E}{f}$, and s for $\frac{g}{f}$, to which they are equal, the other parts being exactly the ſame in both equations.

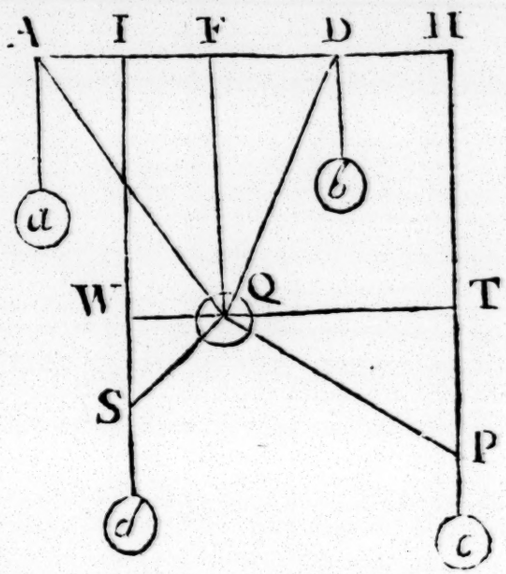
PROBLEM VIII.

AT the fixed points A, D, S, P, ſituated in a vertical plane, wherein ADH is an horizontal line, are placed ſmall pullies over which the threads Q A a, Q D b, Q P c, and Q S d, freely ſlide. To the ends a, b, c, d, of theſe threads, are appended the given weights a, b, c, d, the other ends are faſtened to a given weight Q. Now it is required to determine

the poſition of the weights, when the middle one Q ſuſtains the others in equilibrio.

SOLUTION.

Let the poſition in the figure repreſent the required one. From Q draw QF perpendicular to ADH; draw alſo WQT parallel to AH, and produce *d*S to I, *c*P to H. Put AD $= n$, AI $= m$, DH $= s$, IS $= t$, HP $= r$, A, B, C and D, for the reſpective lengths of the ſtrings Q*A a*, Q*D b*, &c. x for FQ, and y for FD, then $QD = \sqrt{x^2+y^2}$, $Db = B - \sqrt{x^2+y^2}$, $Aa = A - \sqrt{n-y^2+x^2}$, $Sd = D - \sqrt{n-m-y^2+t-x^2}$, and $Pc = C - \sqrt{s+y^2+r-x^2}$; hence, by Lemma 1, the perpendicular diſtance of the common center of gravity of the five weights from the horizontal line AH, will be expreſſed by



$$\frac{aA - a\sqrt{n-y^2+x^2} + bB - b\sqrt{x^2+y^2} + cr + cC - c\sqrt{s+y^2+r-x^2} + dt + dD - d\sqrt{n-m-y^2+t-x^2} + Qx}{a+b+c+d+Q},$$

which, being a maximum, put into fluxions with y conſtant, gives when properly reduced $a \times \frac{FQ}{AQ} - b \times \frac{FQ}{QD} = Q - c \times \frac{TP}{QP} - d \times \frac{SW}{SQ}$, and by putting the ſame expreſſion into fluxions with x conſtant, we ſhall have $a \times \frac{AF}{AQ} + d \times \frac{WQ}{SQ} = b \times \frac{FD}{QD} - c \times \frac{TQ}{PQ}$.

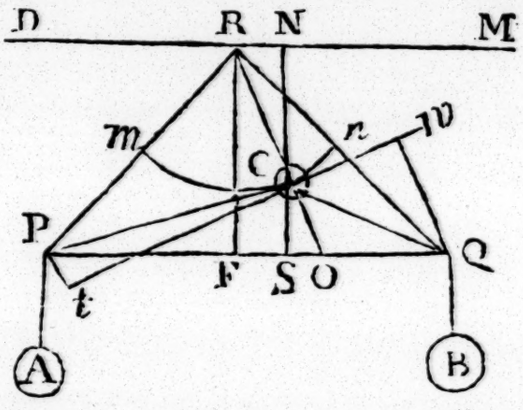
Corol. If d and c are each $= 0$, the firſt equation will become $a \times \frac{FQ}{AQ} + b \times \frac{FQ}{QD} = Q$, and the ſecond $a \times \frac{AF}{AQ} = b \times \frac{FD}{QD}$, which exactly agrees with the ſolution to the 3d Problem.

PROBLEM IX.

SUPPOSE a given weight C, to be ſuſpended by means of a ſtring to a given point R; and ſuppoſing two other given weights A and B to act upon the former by two ſtrings paſſing over two pullies at the given points P, Q, in the ſame horizontal line PQ; 'tis required to find the poſition of each weight when they are in equilibrio.

SOLUTION.

Through the point R, draw a right line DRM parallel to PQ; and from the point C, the ſuppoſed poſition of the given weight C, draw CN perpendicular to DM. Put PR $= a$, RQ $= b$, RC $= d$, L and l for the lengths of the ſtrings CQB, CPA, s and c for the ſine and coſine of the angle PRQ, m and n for the ſine and coſine of the angle QRM, x for the ſine of the angle CRQ, radius $= 1$; then the ſine of PRC will be expreſſed by $s\sqrt{1-xx} - cx$, and its coſine by $c\sqrt{1-xx} + sx$; whence, by Lemma 2d, PC $= \sqrt{a^2+d^2-2adc\sqrt{1-xx}-2adsx}$ and CQ $= \sqrt{bb+dd-2bd\sqrt{1-xx}}$, the ſine of the angle CRN is $nx+m\sqrt{1-xx}$, therefore CN $= dnx + dm\sqrt{1-xx}$; conſequently, the diſtance of the common center of gravity of the three weights from the horizontal line DM will, by Lemma 1, be expreſſed by



$$\frac{Cdnx + Cdm\sqrt{1-xx} + LB}{A+B+C}$$

$$-B\sqrt{b^2+d^2-2bd}\times\sqrt{1+xx}+lA-A\times\sqrt{a^2+d^2-2adc\sqrt{1-xx}-2adsx},$$

$$A+B+C$$

which being put into fluxions, and properly reduced, gives this equation, $C\times\overline{n\sqrt{1-xx}-mx} - B\times\frac{bx}{CQ} = A\times\frac{acx-as\sqrt{1-xx}}{PC}$, or $B\times\frac{bx}{CQ} - C\times\overline{n\sqrt{1-xx}-mx} = \frac{as\sqrt{1-xx}-acx}{PC}\times A$. Now it is very evident, that $\frac{bx}{CQ}$ is the ſine of the angle RCQ; or, if RC be produced, the ſine

of QOC, $n\sqrt{1-xx}-mx$ is the ſine of the angle RCN, or its equal SCO, made by producing NC, as it is eaſy to prove; and laſtly, becauſe $s\sqrt{1-xx}-cx$ expreſſes the ſine of PRC, $\frac{as\sqrt{1-x^2}-acx}{PC}$ will expreſs the ſine of the angle PCR or PCO, therefore $B \times \text{ſin} < QCO - C \times \text{ſin} < SCO = A \times \text{ſin} < PCO$.

This problem was propoſed in a treatiſe called the Mathematician, and there anſwered in the following manner:

Suppoſe the weight C to move in the arch *m n*, towards *m*, with a velocity expreſſed by unity; let RC produced, meet PQ in O, and let CS be perpendicular to PQ; then the velocity of C, in the perpendicular direction CS, being to the abſolute velocity as radius to the coſine of *t* CS, (or the ſine of the angle SCO) will be expreſſed by the ſine of SCO, therefore the momentum of the weight C in the direction perpendicular to the horizon is C × SCO. Moreover, the velocity of C in the direction CP, (or the velocity of A in the direction of the horizon) being as the coſine of *t* PC (or the ſine of POC) will be expreſſed by the ſine of the angle PCO, and the momentum of the weight A towards the horizon will be A × ſine POC. In like manner, the momentum of B from the horizon will be expreſſed by B × ſine of the angle QCO; but when the weights are in equilibrio, the momentum of the two former muſt be equal to that of the latter, or C × ſine SCO + A × ſine PCO = B × ſine QCO, which exactly coincides with the former ſolution.

Corol. 1. If C = o, then A × ſine PCO = B × ſine QCO, which is the caſe of the balance of unequal brachia; but if A = B, then it will become the ſame as the common balance, and RO will biſect PCQ.

Corol. 2. If B = o, then A × ſine PCO = — C × ſine SCO, in which caſe C will be on the other ſide of the perpendicular RF; therefore if A = C, the line CO will biſect the angle PCS.

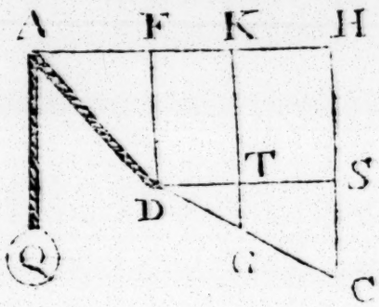
Corol. 3. But if A = o, then C × ſine SCO = B × ſine QCO; therefore when C = B, the angle SCO will be equal to the angle QCO.

PROBLEM

PROBLEM X.

TO determine the position of a beam CD, moveable about one end C as a center, and sustained at the other end D by a given weight Q, appended to a cord QAD, passing over a pulley at a given point A, placed in the horizontal line AH.

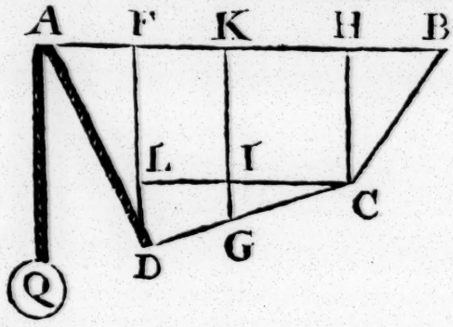
SOLUTION.



Let G be the center of gravity of the beam; also let DF, GK and CH be perpendicular to AH; draw DS parallel to AH. Put $DC = d$, $AH = b$, $HC = p$, $CG = a$, $DG = b$, $AF = x$, the cord $QAD = l$, and the weight of the beam $= w$. Then $CS = \sqrt{dd - \overline{b-x}^2}$, $DF = p - \sqrt{d^2 - \overline{b-x}^2}$, $TG = \frac{b}{d} \times \sqrt{dd - \overline{b-x}^2}$, and $AD = \sqrt{p^2 + d^2 - \overline{b-x}^2 - 2p\sqrt{dd - \overline{b-x}^2} + xx}$; and by Lemma the first, $\frac{Ql - \sqrt{p^2 + d^2 - \overline{b-x}^2 - 2p\sqrt{d^2 - \overline{b-x}^2} + x^2}}{Q+w} \times Q + pw - w\sqrt{d^2 - \overline{b-x}^2} + \frac{bw}{d}\sqrt{d^2 - \overline{b-x}^2}$ is the perpendicular distance of the common center of gravity of the weight and beam from the horizontal line AH, which must be a maximum, in fluxions gives, when the whole is divided by $\dot{x}$, $Q \times \frac{b-x+x-p \times \overline{b-x}}{\sqrt{dd - \overline{b-x}^2}} \div \sqrt{p^2 + d^2 - \overline{b-x}^2 - 2p\sqrt{d^2 - \overline{b-x}^2} + x^2} = w \times \frac{b-d}{d} \times \frac{b-x}{\sqrt{d^2 - \overline{b-x}^2}}$; that is, $\frac{CG \times FH}{DC} \times Q = \frac{HC \times FH - AH \times CS}{AD} \times w$. Now by plane trigonometry, AD : 1 (radius) : : DF : $\frac{DF}{AD}$ = sin $\prec$ DAF, its cos $\frac{AF}{AD}$. The sine of $\prec$ DCS is $\frac{DS}{DC}$, its cosine $\frac{CS}{DC}$, sin $\prec$ ADS is $\frac{DF}{AD}$, that of $\prec$ SDC is $\frac{CS}{DC}$, cosine $\frac{DS}{DC}$; hence $\frac{DF}{AD} \times \frac{DS}{DC} - \frac{CS}{DC} \times \frac{AF}{AD}$, is the sine of the angle ADC. The sine of $\prec$ HCD is $\frac{DS}{DC}$.

therefore the ſine of ∢ ADC is to that of ∢ FDC as $\frac{DF \times DS - CS \times AF}{AD}$ to FH. But $\overline{AF+FH} \times CS - \overline{HS+CS} \times FH$, that is, $AF \times CS - DF \times FH$ is equal to $SC \times AH - HC \times FH$, conſequently $w : Q :: \text{ſin} \measuredangle ADC : \frac{CG}{DC} \times \text{ſin} \measuredangle FDC$.

Mr. *Thomas Simpſon*'s method of ſolution, ſee his Fluxions, *page* 157.



Let C be the center of gravity of the beam; alſo let DEGK, *&c.* be drawn as before. Put $AH = a$, $CH = b$, $CD = c$, $CG = d$, $DL = x$, $CL = y$, and the weight of the beam $= w$. Then $AF = a - y$, $DF = b + x$, and $AD = \sqrt{AF^2+DF^2} = \sqrt{a^2-2ay+yy+bb+2bx+xx}$, which (becauſe $y^2+x^2 = cc$) will alſo be equal to $\sqrt{a^2+b^2+c^2+2bx-2ay} = \sqrt{ff+2bx-2ay}$ (by putting $ff = aa+bb+cc$) whoſe fluxion $\frac{b\dot{x}-a\dot{y}}{\sqrt{ff+2bx-2ay}}$ into Q is the momentum of the weight Q, ſuppoſing the beam to be in motion. Moreover, becauſe $DC : DL :: CG : GI$, we have $GT = \frac{dx}{c}$, whoſe fluxion $\frac{d\dot{x}}{c}$ multiplied by w, is the momentum of the beam itſelf in a vertical direction. Wherefore, making theſe momenta equal to each other (according to the principles of mechanics) we get $\frac{b\dot{x}-a\dot{y}}{\sqrt{ff+2bx-2ay}} = dw\dot{x} \times \sqrt{ff+2bx-2ay}$. But ſince $yy + xx = cc$, we have $2y\dot{y}+2x\dot{x} = 0$, or $-\dot{y} = \frac{x\dot{x}}{y}$; and therefore (by ſubſtitution) $b\dot{x} + \frac{ax\dot{x}}{y} \times cQ = dw\dot{x}\sqrt{ff+2bx-2ay}$, or $\overline{by+ax} \times cQ = dyw \times \sqrt{ff+2bx-2ay}$; from whence and the foregoing equation $xx+yy = cc$, both x and y may be found.

The ſame otherwiſe.

It is evident from mechanics, that the force which acting in the direction DF, would ſuſtain the end D, is to the whole w as CG to CD, and is therefore $= \frac{CD}{CG} \times w$. It is likewiſe known, that two forces acting in

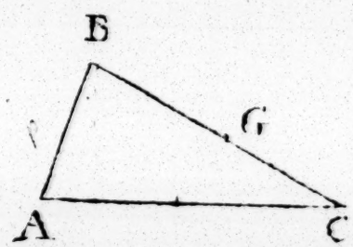
different directions DF and DA, ſo as to have the ſame effect in ſuſtaining DC, or cauſing it to move about the point C, muſt be to each other inverſely, as the ſines of the angles of incidence FDC and ADC; therefore we have ſine FDC : ſine ADC : : Q : $\frac{CD}{CG} \times w$, from which given ratio of the ſines, the angles themſelves may be found, by an algebraic proceſs independent of fluxions.

COROL. If the poſition of CD be ſuppoſed given, and the tenſion of AD (or the weight Q) be required, then, from the foregoing proportion, we ſhall have $Q = \frac{s\,FDC}{s\,ADC} \times \frac{CG}{CD} \times w$; which will alſo expreſs the tenſion of AD, when the end C is ſuſtained by a cord BC, inſtead of a point at C. Whence it follows, that the tenſions of the two cords AD, BC ſuſtaining a beam or rod CD at its extremes D and C, are expreſſed by $\frac{s\,FDC}{s\,ADC} \times \frac{CG}{CD} \times w$, and $\frac{s\,HCD}{s\,BCD} \times \frac{DG}{CD} \times w$; and therefore are to each other as $\frac{CG}{s\,ADC}$ to $\frac{DG}{s\,BCD}$, or $s\,BCD \times CG$ to $s\,ADC \times DG$ reſpectively, becauſe the ſine of FDC, and that of its ſupplement HCD, are equal to each other.

PROBLEM XI.

LET BC be a piece of wood of an indefinite length, moveable about C, as a center; to find the angle BAC, which another piece AB of a given length (p) muſt make with the horizontal line AC, to ſupport the piece CB with the greateſt eaſe, the weight of the piece CB; and CG, the diſtance of the center of gravity from the point C, being given equal to w and a reſpectively.

SOLUTION.



It appears by the laſt problem, that the force acting in the direction AB, neceſſary to ſupport the piece CB at any propoſed elevation above the horizontal line AC, is equal to $\frac{\cos < ACB}{\sin < ABC} \times$

$\frac{CG}{BC} \times w$. Let s and c be the ſine and coſine of the angle BCA, x the ſine of the angle ABC, radius $= 1$; then, by plane trigonometry, we have $s : p :: x : \frac{px}{s} = AC$, and $cx + s\sqrt{1-xx} =$ ſine of the angle BAC, and again, $s : p :: cx + s\sqrt{1-xx} : \frac{p}{s} \times \overline{cx + s\sqrt{1-xx}} = BC$, whence we have $\frac{caw}{\frac{p}{s} \times \overline{cxx + sx\sqrt{1-xx}}}$, for $\frac{\cos \sphericalangle ACB \times CG}{\text{ſin} \sphericalangle ABC \times BC}$, which will be the leaſt poſſible when the denominator is a maximum, in fluxions $2cx\dot{x} + \frac{2ss\dot{x} - 4sx^3\dot{x}}{2\sqrt{ssxx - ssx^4}} = 0$, whence $2c = \frac{2s^2x^2 - s^2}{s\sqrt{x^2 - x^4}}$, and therefore $4xx = 4x^4 + ss$; by compleating the ſquare and proper reduction, we get $x = \sqrt{\frac{c+1}{2}}$; let this value of x be ſubſtituted in $cx + s\sqrt{1-x^2}$, which expreſſes the ſine of BAC, and there will ariſe $c \times \sqrt{\frac{c+1}{2}} + s \times \sqrt{\frac{1-c}{2}}$ for the ſaid ſine or its equal, *viz.* $\frac{c \times \sqrt{c+1} + \sqrt{1-c^2} \times \sqrt{1-c}}{\sqrt{2}}$ (by writing $\sqrt{1-c^2}$ for s) which contracted becomes $\frac{\overline{c+1-c} \times \sqrt{c+1}}{\sqrt{2}}$ or $\frac{\sqrt{1+c}}{\sqrt{2}}$; whence it appears, that the angles at B and A will be equal, let the angle of elevation BCA, length of the ſupporter AB, and diſtance of G from the point C, be what they will.

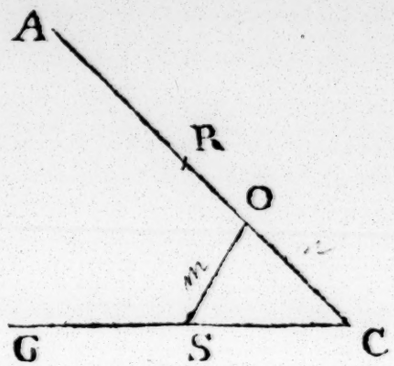
PROBLEM XII.

LET AC be a piece of wood of an indefinite length, moveable about the center C; to find the angle CSO, which another piece OS of a given length *(m)* muſt make with the horizontal line CG, to ſupport the piece AC with the greateſt eaſe; alſo the weight it will ſuſtain, and the value of the angle ACG when the ſaid weight is a maximum, the weight of the piece AC and RC, the diſtance of the center of gravity from the point C being given equal to *n* and C reſpectively.

SOLU-

SOLUTION.

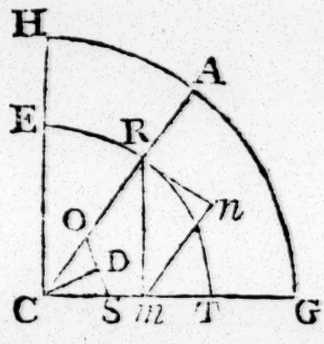
It is evident by the laſt problem, that OS will ſupport CA with the greateſt eaſe, when the triangle SCO is iſoſceles, or SC = OC; and by Problem X, it appears, that the force acting upon the prop SO, will be expreſſed by $\frac{\cos \angle ACS \times CR \times C}{\sin \angle SOC \times CO}$.



Now let x be the ſine of half the angle OCS, to the radius 1; then $\sqrt{1-x^2}$ = cos of $\frac{1}{2} \angle$ OCS, and $2x\sqrt{1-xx}$ = ſin $\angle$ OCS; therefore its coſine is $\sqrt{1-4x^2+4x^4}$. Again, $x : \frac{n}{2} :: 1 : \frac{n}{2x}$ = CO, and conſequently $\frac{\cos \angle ACS \times CR \times C}{\sin \angle SOC \times CO}$, is $n \times \frac{\sqrt{1-4x^2+4x^4}}{\sqrt{1-xx} \times \frac{n}{2x}}$, which muſt be a maximum; therefore the fluxion of the expreſſion $\frac{y-4yy+4y^3}{1-y}$ (wherein $y = \dot{x}x$) being made = 0, we have $8y^3+8y-16yy-1 = 0$, $\therefore y = \frac{1}{2}$; in this caſe it appears, that the angle OCS muſt be 90°, ſo that this value of y gives the minimum; but to find the maximum from hence, let the cubic equation above found be divided by $y-\frac{1}{2}$, and the quotient $8yy-12y+2$ muſt be equal to nothing; and therefore $y = \frac{3}{4} \pm \frac{\sqrt{5}}{4}$ and $x = .4379$, $\therefore$ the angle OCS is 51°. 50′.

Mr. *Simpſon*'s method of ſolution to this problem, ſee his Fluxions, *page* 525.

Suppoſe Rn perpendicular to CR, mn to Rn, and CD to OS; and put the ſine of the angle CRm (= Rmn) = x, radius = 1, and the required weight = y. Then by the laws of mechanics, as Rm : Rn; or 1 : x :: n : (nx) the preſſure, which a power acting at the point R in the direction Rn, muſt ſuſtain to ſupport the beam AC: but as CO : CR (c) :: nx : $\frac{cnx}{CO}$ the force required to ſupport it at the point O, in a perpendicular direction; hence as CD : CO :: $\frac{cnx}{CO}$: $\frac{cnx}{CD}$ the weight ſuſtained by



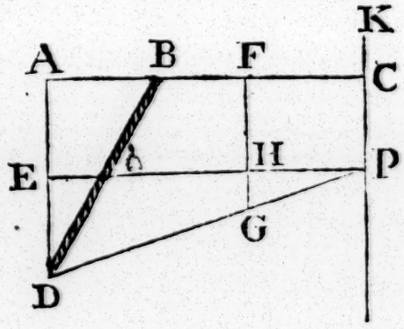
the ſupporter OS, which will be the leaſt, when the perpendicular CD is the greateſt it can be, and that will conſequently be when the angles O and S are equal, let the angle ACG be what it will; but by the elements of trigonometry, $1+x : \sqrt{1-xx} :: 1 : \left(\sqrt{\frac{1-x}{1+x}}\right)$ the tangent of the angle OCD, hence as $\sqrt{\frac{1-x}{1+x}} : \frac{m}{2}$ (OD) $:: 1 : \frac{m}{2}\sqrt{\frac{1+x}{1-x}} =$ CD; therefore $\frac{cnx}{DC}$ is equal to $\frac{2cnx}{m}\sqrt{\frac{1+x}{1-x}} = y$, which put into fluxions, and reduced x, will be found $= 61799$, $=$ the coſine of 51°. 50′. which is the value of the angle ACG, when the required preſſure is the greateſt poſſible.

PROBLEM XIII.

IT is required to find the poſition of the beam DP (whoſe length DP, and place G of its center of gravity are given) when it is ſuſtained by means of a ſtring DB, of a given length faſtened to the end D of the beam, and to a fixed point B; the other end P of the beam leaning againſt a wall KCP, perpendicular to the horizon.

SOLUTION.

From the center of gravity G draw GF parallel to KCP, and from P draw PE parallel to the horizontal line ABFC; upon CD produced let fall from D, the perpendicular DA. Put BC $= a$, BD $= b$, DP $= l$, PG $= m$, DG $= n$, AB $= x$; then AD $= \sqrt{bb-xx}$, and ED $= \sqrt{l^2-\overline{a+x}^2}$, therefore AE $= \sqrt{b^2-x^2} - \sqrt{l^2-\overline{a+x}^2} =$ FH, to this add HG $= m\sqrt{l^2-\overline{a+x}^2}$, we have FG $= \frac{\sqrt{b^2-x^2}}{l} - \frac{n}{l}\sqrt{l^2-\overline{a+x}^2}$; which, being the deſcent of the center of gravity, muſt be the greateſt poſſible, in fluxions $\frac{l\dot{x}\dot{x}}{\sqrt{b^2-x^2}} = \frac{n\times\overline{a+x}\times\dot{x}}{\sqrt{l^2-\overline{a+x}^2}}$; hence $\frac{DP\times AB}{AD} = \frac{AC\times DG}{ED}$, and conſequently AB : AC $:: \frac{DG}{DP} : \frac{ED}{AD}$.



COROL. 1. If BC = o, then $\frac{DG}{DP} = \frac{ED}{AD}$ or $\frac{n}{l} = \frac{\sqrt{l^2-x^2}}{\sqrt{b^2-x^2}}$, $\therefore x = \sqrt{\frac{l^4-a^2b^2}{l^2-n^2}}$.

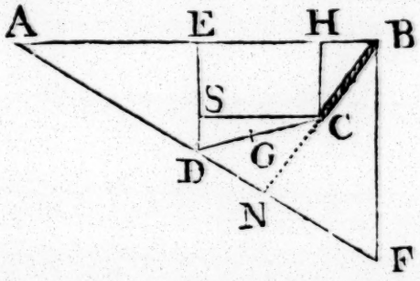
COROL. 2. If DG = GP (BC ſtill = o) then AD = 2 ED.

PROBLEM XIV.

TO determine the poſition of the beam DC, when ſuſtained at reſt by the inclined plane AF, and ſtring BC, faſtened to the beam at C; the other end being moveable (by means of a loop) about the fixed point B.

SOLUTION.

Let AB and FB be drawn, the former parallel, and the latter perpendicular to the horizon, and ſuppoſe DCB to be the poſition required. Draw DE, CH parallel to DF, and SC parallel to AB, produce BC to N. Put AB = a, FB = b, CB = l, DG = (G being the center of gravity of the beam) m, CG = n, s and c for the ſine and coſine of the angle FAB, x = ſine of the angle ANB, radius = 1. Then $x : a :: s : \frac{sa}{x}$ = BN, $\frac{sa}{x} - l$ = NC, and $m + n\,(b) : x :: \frac{sa}{x} - l : \frac{sa-lx}{b}$ = ſin < CDN. $\therefore \sqrt{1-\frac{sa-lx^2}{b}}$ = cos < CDN, whence the ſine of the angle SCD is $\frac{sac-lcx-s\sqrt{b^2-sa-lx^2}}{b}$, and therefore SD = $sac-lcx-s\sqrt{b^2-sa-lx^2}$.



The ſine of the angle ABN is $cx + s\sqrt{1-xx}$, HC = $lcx + ls\sqrt{1-x^2}$, whence the perpendicular diſtance of G, the center of gravity of the beam from AB, is $\frac{nsac+mlcx+hls\sqrt{1-xx}-ns\sqrt{b^2-sa-lx^2}}{b}$, which muſt be a maximum, its fluxion being made equal to nothing, and divided by $\dot{x}$,

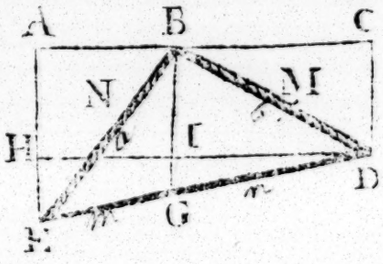
gives $mlc - \frac{blsx}{\sqrt{1-xx}} = ns \times \frac{sa-lx}{\sqrt{b^2-sa-lx^2}}$, from whence the value of x may be determined.

PROBLEM XV.

TO determine the poſition of a beam ED, when ſuſpended at reſt by two ſtrings BE, BD, of given lengths, faſtened to the point B in the horizontal line AC.

SOLUTION.

Suppoſe EBD to be the required poſition; from the points E, D, let fall upon AC, the perpendiculars EA, DC; and from G the center of gravity of the beam, draw GI perpendicular to the line DH, drawn parallel to AC. Put $BE = a$, $BD = b$, $EG = m$, $GD = n$, $m + n = l$, ſin $\angle EBD = s$, cos $= c$, ſin $\angle CBD = z$, radius $= 1$; then, by plane trigonometry, we have $s\sqrt{1-zz} + cz =$ ſin $\angle EBC$ or ABF, whence $1 : a :: s\sqrt{1-zz} + cz : as\sqrt{1-zz} + acz = AE$; and again, $1 : b :: z : bz = CD$, conſequently the perpendicular diſtance of G, the center of gravity of the beam ED from the horizontal line AC, *viz.* $\frac{DG}{ED} \times \overline{AE-DC} + CD$ (for $ED : AE-DC\,(EH) :: DG : (GI)\,\frac{DG}{DE} \times \overline{AE-DC}$) is equal to $\frac{nas\sqrt{1-zz} + nacz - bnz + lbz}{l}$; this by the property of the center of gravity muſt be a maximum, in fluxions $\frac{nas\dot{z}z}{\sqrt{1-zz}} = nac\dot{z} - bn\dot{z} + lb\dot{z}$, whence $\frac{nasz}{\sqrt{1-zz}} = nac - bn + bl$, or $\frac{nasz}{\sqrt{1-zz}} = nac + bm \therefore \frac{z}{\sqrt{1-zz}} = \frac{c}{s} + \frac{bm}{nas}$, put $\frac{c}{s} + \frac{bm}{nas} = p$, then $z = \frac{p}{\sqrt{1+pp}}$

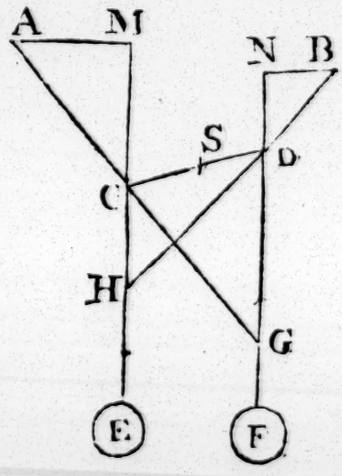


If

If we deſire to have a ſolution to this problem independent of the method of fluxions, then (according to that general principle of mechanics, *viz.* That if a body be ſuſtained at reſt by two ſtrings, as E M, D N, that body when at reſt will have its center of gravity in the right line, which, being drawn perpendicular to the horizon, paſſes through the point where the ſuſtaining ſtrings E M, D N, being produced, interſect) G I being produced to A C, muſt be perpendicular to it in the point of interſection B. This being premiſed, and other things remaining as above, we ſhall have $\sqrt{1-z^2}$ for the ſine of D B G, and becauſe the ſine of A B E is $s\sqrt{1-zz}+cz$, as before ſhewn, its coſine will be expreſſed by $\sqrt{1-ss+sszz-cczz-2scz\sqrt{1-zz}}$ equal to the ſine of E B G. Let the ſine of B G E or B G D $= x$, then $x : a$: : ſine of E B G : E G, and $x : b$: : ſine of G B D : G D; by the former of theſe proportions we have $x = \frac{a \times \text{ſine E B G}}{\text{E G}}$, and by the latter $x = \frac{b \times \text{ſine G B D}}{\text{G D}}$; conſequently, $\frac{b \times \text{ſine G B D}}{\text{G D}} = \frac{a \times \text{ſine E B G}}{\text{E G}}$, that is $\frac{b}{n} \times \sqrt{1-zz} = \frac{a}{m} \times \sqrt{1-ss+sszz-cczz-2scz\sqrt{1-zz}}$.

COROL. If $b = a$, $n = m$, and $s = 1$, then z in the former ſolution, which is there equal to $\frac{p}{\sqrt{1+pp}}$ becomes equal to $\sqrt{\frac{1}{2}}$, which ſhews that the angles C B D, E B A, are each 45°. The reſulting equation above will in this caſe alſo give $1-zz = zz$, becauſe when $s = 1$, $c = 0$, whence z is here equal to $\sqrt{\frac{1}{2}}$, the very ſame as before.

Sir *Iſaac Newton*, at *page* 161, Univerſal Arithmetic, ſays, It is not always neceſſary to ſolve queſtions that are of the ſame kind, particularly by algebra; but from the ſolution of one of them you may moſt commonly infer the ſolution of the other. As if now there ſhould be propoſed this queſtion:



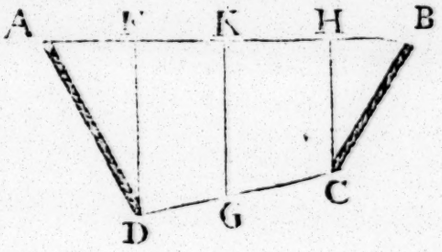
The thread ACDB being divided into the given parts AC, CD, DB, and its ends being faſtened to the two tacks given in poſition A and B; and if at the points of diviſion C and D, there are hanged the two weights E and F; from the given weight F, and the ſituation of the points C and D, to know the weight E.

From the ſolution of the former problem, (*viz.* the 49th in the Univerſal Arithmetic, and the ſame with Problem X. in this treatiſe) the ſolution of this may be eaſily enough found. Produce the lines AC, BD, till they meet the lines DE, CE, in G and H, and the weight E will be to the weight F, as DG to CH. Produce HC and GD indefinitely towards M and N; join the points H, G, and ſuppoſe S the common center of gravity of the two weights E, F, the angle ACM = angle CGD, and BDN = CHD. By plane trigonometry, HC : ſin ∠ CDH :: CD : $\frac{\text{ſin} \angle \text{CDH} \times \text{CD}}{\text{HC}}$ = ſin ∠ BDN, and DG : ſin ∠ GCD :: CD : $\frac{\text{ſin} \angle \text{GCD} \times \text{CD}}{\text{DG}}$ = ſin ∠ ACM, conſequently HC : DG :: $\frac{\text{ſin} \angle \text{CDB}}{\text{ſin} \angle \text{BDN}}$: $\frac{\text{ſin} \angle \text{ACD}}{\text{ſin} \angle \text{ACM}}$: F : E :: CS : SD. Now if we imagine CD to be a beam having its center of gravity at S, let the points A and B coincide, then this particular caſe of the above problem exactly agrees with our's, and the very ſame equation will ariſe by ſubſtituting the ſame letters as before, and expreſſing in thoſe terms the ratio of F to E, or that of CS to SD.

We may otherwiſe arrive at the ſame concluſions, by help of what has been already determined in the ſeveral ſolutions to Problem X. For it is there ſhewn, that the tenſions of two cords AC, BD, (in the above *fig.* where AM and BN are drawn parallel to the horizon) ſuſtaining a beam CD, at its extremes C and D, are to each other as ſin ∠ BDC × DS : ſin ∠ ACD × SC. Let Q and P repreſent the forces acting in the directions CA and DB reſpectively, then CA : AM :: Q : $\frac{\text{AM}}{\text{CA}} \times \text{Q}$ = the force acting in the direction AM, and DB : BN :: P : $\frac{\text{P} \times \text{BN}}{\text{DB}}$ = the force acting in the direction BN; theſe forces, as they act in oppoſite directions, muſt deſtroy each other, otherwiſe the beam CD could not be at reſt; hence Q : P :: $\frac{\text{BN}}{\text{DB}}$: $\frac{\text{AM}}{\text{CA}}$ :: ſin ∠ BDN : ſin ∠ ACM, but Q : P :: ſin ∠ BDC × DS : ſin ∠ ACD × SC, therefore $\frac{\text{SC} \times \text{ſin} \angle \text{NDB}}{\text{ſin} \angle \text{BDC}}$ = $\frac{\text{ſin} \angle \text{ACM} \times \text{DS}}{\text{ſin} \angle \text{ACD}}$, and CS : DS :: $\frac{\text{ſin} \angle \text{CDB}}{\text{ſin} \angle \text{BDN}}$: $\frac{\text{ſin} \angle \text{ACD}}{\text{ſin} \angle \text{ACM}}$; which agrees exactly with the proportion deduced from Sir *Iſaac Newton*'s method, and, the points A, B, coinciding, will give the ſame equations as were obtained by the other ſolutions.

PROBLEM XVI.

LET the beam DC, whoſe center of gravity is in the point G, be ſuſpended by means of two ſtrings DA, CB, faſtened to the ends D, C, of the beam, and to the points A and B in the horizontal line AB; it is required to find the poſition of the ſaid beam when ſuſtained at reſt, ſuppoſing AD=AB, DC=CB, and DG=GC.



SOLUTION.

Imagine the ſituation of the beam in the figure to be the poſition required. From the points D, G, C, draw DF, GK and CH perpendiculars to AB, draw alſo DB. Put $AB = AD = a$, $DC = CB = d$, ſine of the angle $ADF = z$, radius $= 1$. Then $\frac{\sqrt{1+z}}{\sqrt{2}} =$ ſin $\prec$ ABD and $\frac{\sqrt{1-z}}{\sqrt{2}}$ its cos, $\frac{dd-aa+aaz}{dd}$ is the cos $\prec$ DCB, by Lemma 2, $\therefore DB = a\sqrt{2-2z}$, whence the ſine of $\prec$ DCB is $\sqrt{2a^2d^2 \times 1-z - a^4 \times 1-z^2}$, and $a\sqrt{2-2z} : d :: \frac{\sqrt{2a^2d^2 \times 1-z - a^4 \times 1-z^2}}{dd}$ $: \frac{\sqrt{2dd - a^4 \times 1-z}}{d \times \sqrt{2}} =$ ſin $\prec$ DBC, therefore we have $\frac{a\sqrt{1-z}}{d\sqrt{2}} =$ cos $\prec$ DBC, and the ſine of $\prec$ HBC, or its equal ADC, is $\frac{a\sqrt{1+z}}{d\sqrt{2}} \times \frac{\sqrt{1-z}}{\sqrt{2}}$ $+ \frac{\sqrt{1-z}}{d\sqrt{2}} \times \frac{\sqrt{2d^2 - a^2 \times 1-z}}{\sqrt{2}}$, conſequently $HC = \frac{\sqrt{1+z} \times a \times \sqrt{1-z}}{\sqrt{2}}$ $+ \frac{\sqrt{1-z} \times \sqrt{2d^2 - aa \times 1-z}}{\sqrt{2}}$, and $\frac{FD+HC}{2}$ or $KG = \frac{3a\sqrt{1-z^2} +}{4}$ $\frac{\sqrt{2d^2 - a^2 + 2aaz - 2ddz - a^2z^2}}{4}$; this being the perpendicular diſtance of the center of gravity of the beam from the horizontal line AB, muſt be a maximum, which being put into fluxions and properly reduced, gives $\frac{3az}{\sqrt{1+z}} = \frac{a^2 - d^2 - aaz}{\sqrt{2dd - 2aa + aaz}}$; from whence, by having a and d given, the value of z may be found by the ſolution of the cubic equation,

$9na^2z^2 + 8a^4z^2 = m^2 - 2ma^2z + a^4z^2 + m^2z - 2m^2z^2$, wherein m is put for $aa - dd$, and n for $2dd - aa$.

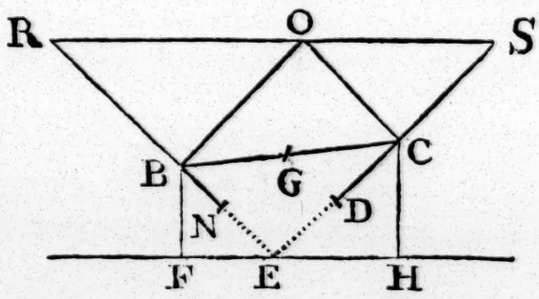
COROL. If $a = d$, then $z = 0$, conſequently ABCDA will become a right angled parallelogram; in this circumſtance, no alteration in the poſition can enſue, by changing the place of the center of gravity of the beam from the middle point G to any other point thereof.

PROBLEM XVII.

LET NBR and DCS be two ſmooth planes inclined to the horizon in given angles; between theſe planes it is required to place a given beam BC, ſo that it ſhall remain at reſt in that poſition.

SOLUTION.

Let BC be the required poſition of the beam, G its center of gravity. Produce BN, CD, until they meet in E, through E draw the horizontal line HEF; and from B, C, let fall thereon the perpendiculars BF, CH. Put $s = \text{ſin} < FEB$, $a = \text{ſin} < CEH$, $m = \text{ſin} < BEC$, $n =$ its cos, radius $= 1$, $l = BC$, $p = BG$, $q = GC$, and $x = BE$, then $\frac{mx}{l} = \text{ſin} < BCE$, and $\frac{\sqrt{l^2 - m^2x^2}}{l}$



$= \text{cos} < BCE$, therefore $\frac{m\sqrt{l^2 - m^2x^2} + nmx}{l} = \text{ſin} < EBC$, hence EC $= nx + \sqrt{l^2 - m^2x^2}$, and $HC = a\sqrt{l^2 - m^2x^2} + anx$; conſequently, $pa\sqrt{l^2 - m^2x^2} + panx + qsx$ expreſſes the perpendicular diſtance of the center of gravity G of the beam BC, from the horizontal line HEF, which (becauſe it muſt be a maximum) being put into fluxions and properly reduced, gives $x = \frac{rl}{\sqrt{p^2a^2m^4 + r^2m^2}}$ (r being put for $qs + pna$) whence EC $= \frac{lpam^2 + nrl}{\sqrt{p^2a^2m^4 + r^2m^2}}$, and $BE : EC :: r : pam^2 + nr$.

COROL.

COROL. 1. When BEC is a right angle, $m = 1$, and $n = 0$, conſequently BE : EC : : $r : pa$, or $qs : pa$.

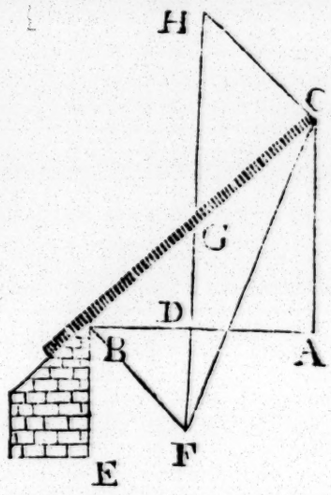
COROL. 2. If the angles FEB, HEC, are equal, and BEC ſtill a right angle, we ſhall have $s = a$, and therefore BE : EC : : $q : p$.

COROL. 3. If $q = p$, or the center of gravity in the middle of BC, other things remaining as in the problem, we ſhall have BE : EC : : $s + na : am^2 + ns + n^2 a$.

COROL. 4. If from the points B, C, we erect the perpendiculars BO, CO, to ER and ES reſpectively; and through their point of interſection O, and the center of gravity G of the beam, we draw a right line, it will (when BEC is a right angle) be perpendicular to the horizon; for in that circumſtance, BE : CE : : $qs : pa$, by Corol. 1, and therefore OC : CE : : $qs : pa$, and $\frac{BO}{BE} \times \frac{BG}{GC} = \frac{qs}{pa} \times \frac{p}{q} = \frac{s}{a}$. Through O draw ROS parallel to the horizontal line HEF, then will the angle SOC be equal to the angle ECH or BEF. Imagine now the planes ABR, DCS, taken away, and the beam BC ſupported ſtill in the ſame poſition by the ſtrings BO, CO; then by the ſolution to the laſt problem but one, the ſine of the angle SOC will be expreſſed by $\frac{s}{\sqrt{s^2+a^2}}$, that is, by s, becauſe $s^2 + a^2 = 1$; conſequently the line joining the points O and G, is perpendicular to the horizon, and therefore no alteration in the poſition of the beam will enſue, if the ſupporting planes are taken away, and in their room the ſtrings BO, CO, are made uſe of to ſuſtain the beam, provided the angle BEC be a right one.

Mr. *Emerſon*, at page 71. of the ſecond edition of his treatiſe on Mechanics, has conſidered this problem, where he tells us, That if any body whatever, as BC, or any beam loaded with a weight, be ſupported by two planes AB, CD, at C and B, the lines CO, BO, be drawn perpendicular to theſe planes; and from the interſection O, the line OK be drawn perpendicular to the horizon, it will paſs through the center of gravity G of the body. This is certainly true, when the angle in which the planes are inclined to each other is a right one, as has been proved above; but as Mr. *Emerſon* ſeems to infer, that this property is general, the angle of the inclination of the planes being aſſumed at pleaſure, it may not be improper to ſhew, that it cannot poſſibly hold true, when the angle BEC is either obtuſe or acute, unleſs in ſome very particular circumſtances; ſuch as

when the angles FEB and HEC are equal, or when BG is equal to GC; becauſe in both theſe, p and q in the expreſſion for the ſine of the angle SOC will intirely vaniſh; but otherwiſe the value of that ſine, which is conſtantly the ſame with that of ECH (a given quantity) will vary according to the different aſſumptions for the place of the center of gravity of the beam; which being manifeſtly impoſſible, it follows, that the determinations in Mr. *Emerſon*'s 53d propoſition and its corollaries, are not univerſally true, nor indeed are thoſe in the 62d, 63d and 65th propoſitions; becauſe as their demonſtrations depend upon the 53d propoſition, they are certainly liable to the above-mentioned objections. In the 64th propoſition, it is ſaid, That if a heavy beam, or one bearing a weight, be ſuſtained at C, and moveable about a point C, whilſt the other end B lies upon the wall BE; and if HGF be drawn through the center of gravity G, perpendicular to the horizon, and BF, CH, perpendicular to BC and CF be drawn, then



The weight of the beam	}	HF
Preſſure at B	}	HC
Force acting at C	}	CF

are reſpectively as and in theſe directions.

That the whole weight is to the preſſure at B, as HF to HC, is true, and may be thus proved: Let W be put for the weight of the beam BC, Q for the force acting in the direction BF, (HC); then, by Problem X, $W : Q :: 1 : \frac{CG}{BC} \times s < BCA$, radius being unity. The triangles BCA, BGD, BGF and GHC, are ſimilar to each other, whence BG : GC :: FG : GH, therefore BC : GC :: FH : GH; again, DA : GC :: HC : GH by ſimilar triangles, conſequently BC : DA :: FH : HC. BC : BA :: 1 : ſin $<$BCA, and BC : CG :: BA : DA; multiply extremes and means, we get $\frac{CG}{BC} = \frac{DA}{BA}$, hence $BC : DA :: 1 : \frac{CG}{BC} \times \text{ſin} < BCA$; and becauſe BC : DA :: FH : HC, therefore W : Q :: FH : HC, or the whole weight, to the preſſure at B, as FH to HC.

If by a force acting at C in the direction CF, our author means, that, inſtead of the beam BC being moveable about the point C, it is to be ſupported by two forces acting in the directions BF (HC) and FC reſpectively, then the propoſition will be no longer true, as may be proved without much difficulty by Problem XXVIII.

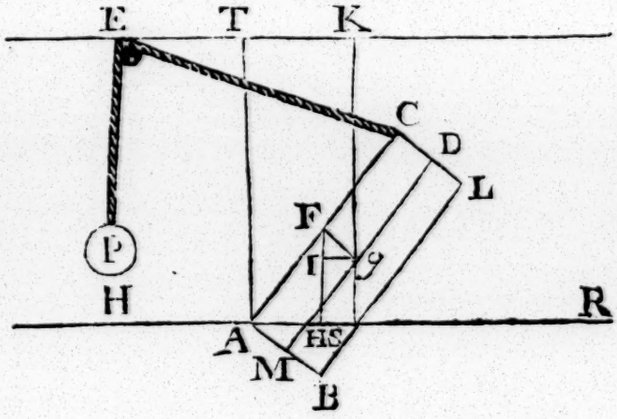
Upon ſuppoſition that the beam BC moves freely upon the fixed point C; and that inſtead of lying upon the inclined plane at B, the end B laid upon the horizontal plane AB, then the weight and the preſſure at B will be as BC to CG. For, by Problem X, W : Q : : ſin ∠ BCA : $\frac{CG}{BC}$ × ſin ∠ BCA, or W : Q : : BC : CG.

PROBLEM XVIII.

LET ABLCA repreſent a right angled cylinder, moveable about a fixed point A, in the circumference of its baſe; and ſuppoſe this cylinder held in a poſition inclined to the horizontal line HR, by means of a ſtring CEP (paſſing freely over a ſmall pulley at E) one end of which is faſtened at the point C, and to the other end a weight P is appended. It is required to determine, by the following data, the angle CAR, when the weight P, and cylinder ABLCA, ſuſtain each other in equilibrio.

SOLUTION.

Let EK be drawn parallel to HR, S*g*K and AT perpendicular to HR, the former paſſing through the center of gravity of the cylinder; from *g* draw *g*F parallel to AB; draw alſo FH perpendicular to HR, *g*I parallel to it, and M*g*D parallel to AC; let there be given AF = FC = M*g* = *a*, F*g* = AM = MB = *b*, AE = *m*, SK = *b*, the ſtring CEP = *l*, weight of the cylinder = *w*, ſin ∠ EAS = *s*, cos = *c*, rad. = 1. Put *x* = ſin ∠ FAH,



then $b \times \sqrt{1-x^2}$ = FI, and $ax - b \times \sqrt{1-x^2}$ = HI = S*g*, whence $b - ax + b\sqrt{1-x^2}$ = K*g*. But by the elements of plane trigonometry, the product of the ſines of any two angles, added to that of their coſines, is equal to the coſine of their difference multiplied by radius, therefore $sx + c\sqrt{1-x^2}$ = cos ∠ EAC. By Lemma 2, $\sqrt{m^2+4a^2-4masx+4amc\sqrt{1-x^2}}$

$= CE$, whence $\frac{Pl - \sqrt{m^2 + 4a^2 - 4mas x - 4mac\sqrt{1-x^2}} \times P + bw - awx}{P+w}$ $\frac{+ bw\sqrt{1-x^2}}{P+w}$ is the perpendicular diſtance of the common center of gravity of the weight and cylinder from EK (by Lemma 1.) which muſt be a maximum; in fluxions we have $-\sqrt{4mas\dot{x} + \frac{4amcx\dot{x}}{\sqrt{1-xx}}} \times P = \sqrt{\frac{bx\dot{x}}{\sqrt{1-xx}}} \times W + aw\dot{x} \times EC$, therefore $2amc \times \tan \measuredangle FAH \times P - 2ams\, P = b \times EC \times \tan \measuredangle FAH \times w - aw\, EC$, and $\tan \measuredangle FAH = \frac{2mas\, P - aEC\, w}{2amc\, P + bEC\, w}$, $= \frac{KS \times AC \times P - EC \times AF \times w}{AC \times ET \times P + Fg \times EC \times w}$.

Corol. 1. If $KS = AC$, and $ET = 0$, or the pulley E placed perpendicularly over A the center of motion, the tangent of $\measuredangle FAH$ will then be equal to $\frac{AC^2 \times P - EC \times AF \times w}{Fg \times EC \times w}$.

Corol. 2. If the propoſed cylinder is to be ſuſtained by the weight P in a direction parallel to the horizon, in ſuch ſort that EC ſhall be at right angles with AC, then $KS \times AC \times P - EC \times AF \times w = 0$; now becauſe EC and KS muſt in this caſe be equal, and AF being conſtantly half AC, it follows that $P = \frac{w}{2}$.

Corol. 3. If the propoſed cylinder is to be ſupported in a poſition perpendicular to the horizon by the weight P, then as the angle FAH muſt be 90, and conſequently its tangent infinite, therefore (becauſe ET $= EC$) $AC \times P = Fg \times w$, and $P : w :: Fg : AC$, a well known property of the bended leaver.

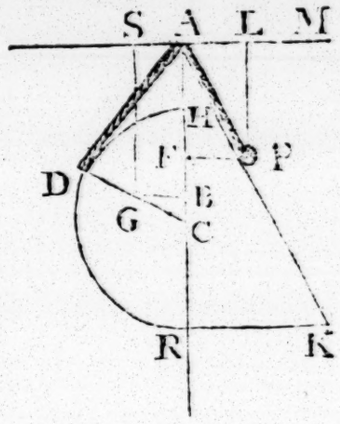
PROBLEM XIX.

SUppoſing a beam CD, moveable about one end C as a center, to be ſuſtained at the other end D by means of a given weight P, hanging at a ſtring paſſing over a pulley at a given point A, vertical to C, and ſupported by an inclined plane APK; it is propoſed to find the poſition

ſition of the beam and weight when they are a juſt counterpoiſe to each other.

SOLUTION.

Becauſe the weight P is to be a juſt counterpoiſe to the beam CD, it is evident, that one and the ſame weight which, (hanging freely) would ſupport DC in the required poſition, would alſo ſupport the weight P at reſt upon the inclined plane AK. Let Q denote that weight, and put W for the weight of the beam; then, by Problem X, we have $Q = W \times \frac{GC}{DC} \times \frac{\text{ſin} \sphericalangle DCR}{\text{ſin} \sphericalangle ADC}$, and by Problem III, we have P : Q : : AK : RA (G being the center of gravity of CD, and ARK a right angle) whence $Q = P \times \frac{RA}{AK}$, therefore $W \times \frac{CG}{DC} \times \frac{\text{ſin} \sphericalangle DCR}{\text{ſin} \sphericalangle ADC} = P \times \frac{RA}{AK} = W \times \frac{CG}{CD} \times \frac{AD}{AC}$. Now put $DC = CH = CR = r$, $AK = s$, $AR = b$, $AC = h$, $CG = m$, $GD = n$, l for the length of the ſtring DAP, and x for the part AP thereof; then the above equation expreſſed in algebraic terms will be $W \times \frac{m}{r} \times \frac{l-x}{h} = P \times \frac{b}{s}$, whence $x = \frac{sml\,W - rhb\,P}{sm\,W}$.



This queſtion may be otherwiſe anſwered, without referring to the problems above-mentioned, by firſt finding the perpendicular diſtance of the common center of gravity of the beam and weight from the horizontal line MAS, drawn through the point A, and then making the fluxion thereof equal to nothing. In order to which, draw PL and GS parallel to the vertical line RCA, PF and GB perpendicular to it; by Lemma 2, the coſine of the angle DCA will be expreſſed by $\frac{rr + hh - \overline{l-x}^2}{2rh}$, GS by $b - \frac{rrm + hhm - m\overline{l-x}^2}{2rh}$, and AF = LP by $\frac{bx}{s}$; conſequently, $\frac{SG \times W + AF \times P}{W+P}$, or $b \times W - \frac{Wmrr + Wmhh - Wm\overline{l-x}^2}{2hr \times \overline{W+P}} + \frac{bx\,P}{s \times \overline{W+P}}$, is, by Lemma 1, the ſaid perpendicular diſtance required, which, being

put into fluxions, gives $P \times \frac{b\dot{x}}{s} = \frac{m W l \dot{x} - m W x \dot{x}}{b r}$, whence $x = \frac{s m l W - r b b P}{s m W}$, the very ſame as before.

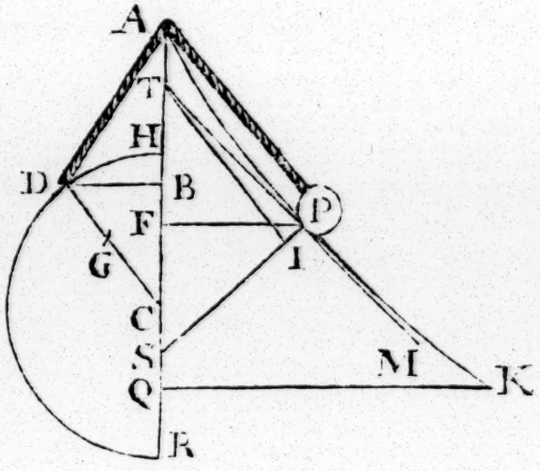
COROL. If AK coincides with AR, then $s = b$, and x becomes equal to $\frac{m l W - r b P}{m W}$, or $l - \frac{r b}{m} \times \frac{P}{W}$.

PROBLEM XX.

SUppoſing a beam CD, moveable about one end C as a center, to be ſuſtained at the other end D, by means of a given weight P, hanging at a rope paſſing over a ſmall pulley at a given point A, vertical to C; it is propoſed to find the curve APK, which the weight muſt aſcend or deſcend, ſo as to be every where a juſt counterpoiſe to the beam.

SOLUTION.

Becauſe the beam and weight are to counterbalance each other, it is evident, that the ſame weight (Q) which being ſuſpended perpendicular to the horizon, will be ſufficient to ſuſtain the beam DC in any poſition, will alſo retain the weight P, in the correſponding poſition, at reſt upon the curve. Let TPM be a tangent to this curve at the point P, FT the ſubtangent, and PS the normal line; from T draw TI parallel to AP, and put W for the weight of the beam DC, its length equal to r, $CG = n$, (G being the place of the center of gravity of the beam) $DAP = l$, AF $= x$, $FP = y$, then will $FT = \frac{y\dot{x}}{\dot{y}}$, $FS = \frac{y\dot{y}}{\dot{x}}$, and $AS = \frac{y\dot{y} + x\dot{x}}{\dot{x}}$, but $AP^2 = xx + yy$, therefore the ratio of Q to P, which is that of TI to TS (by Problem III.) or of AP to AS, by the ſimilar triangles APS, TIS, becomes $\sqrt{x^2+y^2}$ to $\frac{y\dot{y}+x\dot{x}}{x}$, whence $Q \times \overline{y\dot{y}+x\dot{x}} = P \times \dot{x} \sqrt{\overline{x^2+y^2}}$,



and $Q \times \frac{y\dot{y} + x\dot{x}}{\sqrt{x^2+y^2}} = P\dot{x}$; by taking the fluent of each ſide the equation, we get (ſuppoſing no correction neceſſary) $Q \times \sqrt{x^2+y^2} = Px$. Now with regard to the beam CD, we have, by Problem X, $Q : W :: \frac{CG}{CD} \times$ ſin $\sphericalangle$ ACD : ſin $\sphericalangle$ ADC, that is $Q \times \frac{AC}{AD} = W \times \frac{CG}{DC}$, which expreſſed algebraically, becomes $Q \times \frac{m}{l - \sqrt{x^2+y^2}} = W \times \frac{n}{r}$, exterminate Q; by help of theſe equations, there will ariſe $mr\,Px = nl\,W\sqrt{x^2+y^2} - nW \times \sqrt{x^2+y^2}$, or $mr\,Px = nl\,Wz - n\,W\,z^2$ (putting $zz = xx + yy$) the equation of the required curve.

Mr. *Simpſon*'s method of ſolution, ſee *page* 293 of his treatiſe on Fluxions.

From the center C, with the radius CD, let a ſemicircle HDR be deſcribed, and let DB and PF be perpendicular to the vertical line AHCR; alſo let $CD = a$, $CA = b$, $AH = c$, $AF = x$, $PF = y$, $HB = z$, and the length of the rope $DAP = m$; likewiſe let AQ $(= b)$ be the given value of x (AF) when D coincides with H.

Becauſe the weight and beam are always in equilibrio, by the hypotheſis, their momenta, and conſequently their velocities in a vertical direction, muſt be every where in a conſtant ratio; and therefore the diſtance QF $(b - x)$ aſcended by the weight P, will be to the diſtance HB, deſcended by the end of the beam D, likewiſe in a conſtant ratio; let this ratio be that of b to any given quantity d, that is, let $b - x : z :: b : d$, and we ſhall have $db - dx = bz$. Moreover, we have AD^2 $(CD^2 + AC^2 - 2\,AC \times BC) = a^2 + b^2 - 2b \times \overline{a - z} = \overline{b - a}^2 + 2bz = c^2 + 2bz = c^2 - 2db + 2dx$; whence AP $(m - AD) = m - \sqrt{c^2 - 2db + 2dx}$, and therefore $y^2\,(AP^2 - AF^2) = \overline{m - \sqrt{c^2 - 2db + 2dx}}^2 - x^2$.

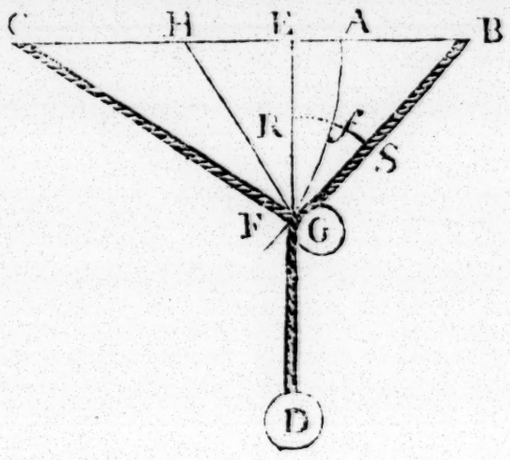
PROBLEM XXI.

LET G be a given weight, hanging freely from the end of a cord CF, faſtened to the point C; and let D be another given weight, ſuſpended by the cord DFB, paſſing over a ſmall pulley at E, which cord

is faſtened in B. It is required to find in what places the weights G and D will ſettle or come to reſt, BC being ſuppoſed an horizontal line.

SOLUTION.

Put the given quantities CF $= a$, CB $= c$, the length of the cord DFB $= b$. Draw EF perpendicular to the horizontal line BC, and put CE $= x$; then EF $= \sqrt{aa-xx}$, FB $= \sqrt{aa+cc-2cx}$, and DEF $= b - \sqrt{aa+cc-2cx} + \sqrt{aa-xx}$, conſequently $\frac{DE\times D+FE\times G}{D+G}$; which, by Lemma 1, is the perpendicular diſtance of the common center of gravity of the two weights D, G, from the horizontal line BC, algebraically expreſſed, and put into fluxions, will produce this equation, $\frac{D\times c\dot{x}}{\sqrt{aa+cc-2cx}} = \frac{\overline{D+G}\times x\dot{x}}{\sqrt{aa+xx}}$, whence (by putting $n = DD$, $m = \overline{D+G}^2$) we have $n\,cc\,aa - n\,c^2\,xx = m\,a^2\,xx + m\,c^2\,xx - 2\,m\,c\,x^3$. If G $= o$, then $n = m$, and our equation becomes $cc\,aa - cc\,xx = a^2\,xx + 2\,c^2\,xx - 2\,m\,c\,x^3$, or $2\,c\,x^3 - 2\,cc\,xx - a^2\,x^2 + a^2\,c^2 = o$; divide by $x-c$, there will ariſe $2\,c\,xx - aax - aa\,c = o$; one root of which will expreſs CE ſuch, that the perpendicular ED paſſes by the pulley F, and the center D of the weight, when the ſame is at reſt. The marquis *de l'Hoſpital*, at *page* 51 of his treatiſe *Des Infiniment Petits*, gives (when G $= o$) the following ſolution:



Call EF, y; BF, z; then will $b - z + y =$ maximum, and ſo $\dot{y} = \dot{z}$. Now it is evident, that the pulley at F does deſcribe a circle CFA, about the point C as a center; and if fR be drawn from the point f, infinitely near to F, parallel to CB, and fS perpendicular to BF, therefore will FR $= \dot{y}$, and FS $= \dot{z}$, which are conſequently equal to each other; and ſo the little right angled triangles FRf, FSf, having the common hypothenuſe Ff, are equal and ſimilar; whence the angle RFf is equal to the angle SFf, that is, the point F muſt be ſo ſituate in the periphery FA, that the angles made by the right lines EF, FB, with the tangents in F, be equal to each other; or elſe (which comes to the ſame) the angles BFC, DFC, equal.

This

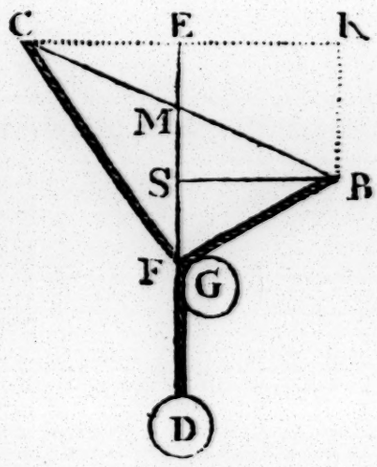
This being premiſed, if you draw FH ſo, that the angle FHC be equal to the angle CFB or CFD, the triangles CBF, CEH, will be ſimilar, as alſo the right angled triangles ECF, EFH; ſince the angle CFE is equal to the angle FHE, each of them being the complement of the equal angles FHC, CFD, to two right angles, and conſequently $CH = \frac{aa}{c}$, and $HE \left(= x - \frac{aa}{c}\right) : EF\ (y) : EF\ (y) : : EC\ (x)$; whence $xx - \frac{aax}{c} = yy = aa - xx$, from the nature of the circle, from whence ariſes the ſame equation as above.

PROBLEM XXII.

OTHER things remaining as in the laſt problem, let the points C and B be ſituated out of the ſame horizontal line; to determine the poſition of the weights G and D, when they ſuſtain each other in equilibrio.

SOLUTION.

Draw CK and BK, the former parallel, and the latter perpendicular to the horizon. Draw alſo FE parallel to KB, and BS to CK. Put $FC = a$, $CB = c$, $CK = m$, $KB = n$, b for the length of the cord DFB, $CE = x$; then $EK = SB = m - x$, $EF = \sqrt{aa - xx}$, and $SF = \sqrt{a^2 - x^2} - n$, conſequently $FB = \sqrt{a^2 - x^2 + n^2 - 2n\sqrt{a^2 - x^2}} + \overline{m - x^2}$, and $ED = \sqrt{a^2 - x^2} + b - \sqrt{a^2 + n^2 - x^2 - 2n\sqrt{a^2 - x^2}} + \overline{m - x^2}$, hence $\frac{DE \times D + EF \times G}{D + G}$; which, by Lemma 1, is the perpendicular diſtance of the common center of gravity of the two weights D, G, from the horizontal line CK, being expreſſed in terms of x and given quantities, put into fluxions and properly reduced, will give $\overline{D + G} \times \dot{x} = \frac{D m \sqrt{aa - xx} - D n x}{\sqrt{a^2 + n^2 + m^2 - 2mx - 2n\sqrt{a^2 - x^2}}}$, whence $\overline{D+G} \times CE \times FB = D \times CK \times FE - D \times BK \times CE$. If $G = o$, our equation becomes $D \times CE \times FB$



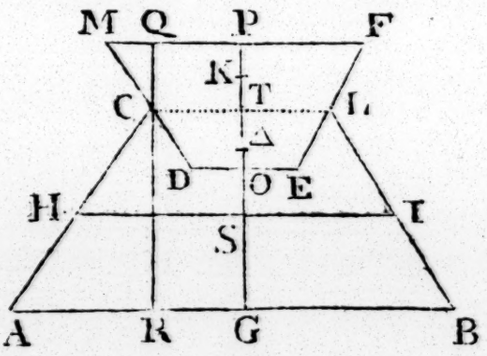
$= D \times CK \times FE - D \times BK \times CE$, or $\overline{FB + BK} \times CE = CK \times FE$; in which, if $BK = o$, or the points C, B, in the ſame horizontal line, we ſhall have $FB \times CE = CK \times FE$.

PROBLEM XXIII.

LET ACMFLBA repreſent the middle ſection, made by a vertical plane, cutting the pyramidal fruſtum MDEFM, and the equal parallelopipedons AC, BL, which (being moveable about the fixed points A, B, in the ſame horizontal line) ſupport the fruſtum, by leaning againſt the ſides MD, FE, thereof. It is required to determine the angle CAB, or its equal LBA, when the ſupporters and propoſed fruſtum are in equilibrio.

SOLUTION.

Biſect MF and DE in the points P and O; draw POG, which will be perpendicular to the horizontal line AB. Join the points C, L, with the right line CLT; let the middle points H, I, or centers of gravity of the ſupporters AC, BL, be joined by the right line HI; then S, the point where it interſects the perpendicular POG, will be the place of their common center of gravity. Through C draw QCR parallel to PSQ, and let K be the place of the center of gravity of the fruſtum. Put $AC = BL = a$, $AB = b$, $MP = PF = c$, $PO = e$, $KO = r$, s for the ſine of the angle FMD, radius $= 1$; the weight of $AC = f =$ that of BL, w for the weight of MFEDM; and let x be the ſine of the angle CAB. Then $ax = CR = TG$, $\frac{ax}{2} = SG = ST$, and $1 : a :: \sqrt{1-xx} :$ $a\sqrt{1-xx} = AR \therefore b - 2a\sqrt{1-xx} = CL$, $c - \frac{b}{2} + a\sqrt{1-xx} =$ MQ, $\sqrt{1-ss} : s :: c - \frac{b}{2} + a\sqrt{1-xx} : QC = \frac{sc - \frac{bs}{2} + as\sqrt{1-xx}}{\sqrt{1-ss}}$ $= PT$, let $m = \frac{s - \frac{bs}{2}}{\sqrt{1-ss}}$, and $n = \frac{as}{\sqrt{1-ss}}$, then $PT = m + n\sqrt{1-xx}$



TO $= e - m - n\sqrt{1-xx}$, and KT $= r - e + m + n\sqrt{1-xx}$, whence KS $= \frac{ax}{2} + r - e + m + n\sqrt{1-xx}$. But $2f + w : 2f :: KS : \Delta K$, ($\Delta$ being the place of the common center of gravity of the three given bodies, *viz.* the two ſupporters and the fruſtum) equal to $\frac{2fr + 2fm - 2fe}{2f+w} + \frac{fax + 2fr\sqrt{1-xx}}{2f+w}$, and therefore $\Delta G = \frac{fax + wax + wr - wc - wm}{2f+w} + \frac{wn\sqrt{1-xx}}{2f+w}$; which, being the diſtance of the common center of gravity from the horizontal line AB, muſt be a minimum; in fluxions we have $fa\dot{x} + wa\dot{x} - wn \times \frac{2a\dot{x}}{2\sqrt{1-xx}} = 0$, or $fa + wa = \frac{wnx}{\sqrt{1-xx}}$; and by reſtoring the value of n, we get $\frac{f+w}{ws} \times \sqrt{1-ss} = \frac{x}{\sqrt{1-xx}}$, that is, $\frac{f+w}{w} \times \tan \sphericalangle MCQ = \tan \sphericalangle CAB$, or its equal LBA.

COROL. 1. If the angles ACD, BLE, are right ones, then the tan $\sphericalangle$ MCQ $=$ tan $\sphericalangle$ CAB, and conſequently $f + w$ becomes equal to w, and therefore $f = 0$; which ſhews, that when the ſupporters AC, BL, are perpendicular to the ſides MD, EF, of the fruſtum, they will (if ſufficiently ſtrong, and the points A and B do not give way) ſupport the ſolid fruſtum MFEDM in that poſition, let the weight thereof be what it will.

COROL. 2. If $f = w$, then the tangent of the angle CAB will be double the tangent of the angle MCQ.

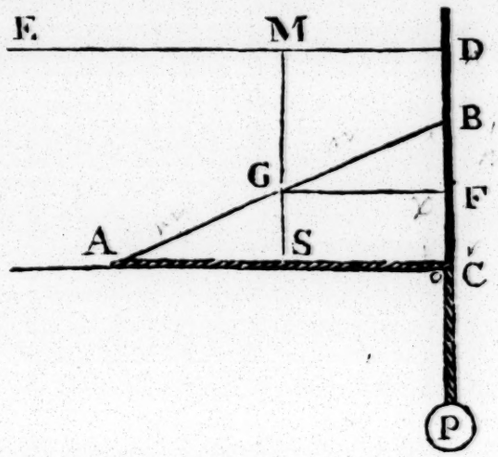
PROBLEM XXIV.

IF AC be a horizontal line upon the point C of which ſtands an upright parallelopipedon CD, one of the plane ſurfaces of which is perpendicular to AC; it is required to find the angle CAB, in which a long ſolid AB (whoſe end B is ſupported by the plane CD) will be ſuſtained in equilibrio by means of a weight P, appended to the end P of a ſtring

DCA, paſſing freely over a ſmall pulley fixed in the point C, whoſe other end is faſtened to the extremity A of that ſolid.

SOLUTION.

From any point D in CD, draw DE parallel to the horizontal line AC; and from G, the center of gravity of the beam or ſolid AB, let SGM, GF, be drawn parallel to CD and AC reſpectively. Put $AG = m$, $GB = n$, $m+n = d$, W for the weight of AB, l for the length of the ſtring ACP, $b = CD$, and $x = BC$, then we have $d : x :: m : \frac{mx}{d} = GS$,



whence $MG = b - \frac{mx}{d}$, $AC = \sqrt{dd-xx}$, $CP = l - \sqrt{dd-xx}$, and therefore $DP = b+l-\sqrt{dd-xx}$, conſequently $\frac{MG \times W + DP \times P}{W+P}$, or $\frac{\overline{b-\frac{mx}{d}} \times W + \overline{b+l-\sqrt{dd-xx}} \times P}{W+P}$, which, by Lemma 1, is the perpendicular diſtance of the common center of gravity of the ſolid AB, and weight P from the horizontal line DE, muſt be a maximum; in fluxions $-\frac{mW\dot{x}}{d} + \frac{P \times x\dot{x}}{\sqrt{dd-xx}} = 0$, whence $\frac{x}{\sqrt{dd-xx}} = \frac{m}{d} \times \frac{W}{P} = \tan \sphericalangle BAC$, to the radius d.

Corol. If $W = P$, and $m = n$, then $\frac{x}{\sqrt{dd-xx}} = \frac{1}{2}$, and the angle $BAC = 26°.\ 34'$.

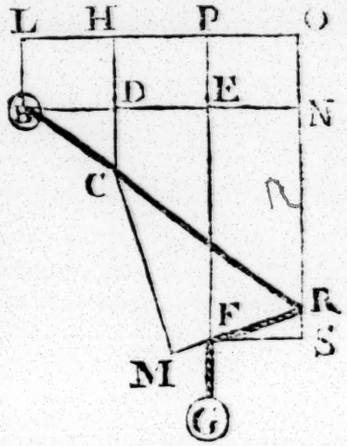
PROBLEM XXV.

LET RCB be a perfect cylindrical inflexible rod, or leaver of the firſt kind (commonly called a Vectis) moveable about a fixed point C; to the end B of this rod a given weight B is faſtened, and to the other end R a ſtring of a given length, paſſing over a ſmall pulley placed at a certain

certain point F. It is required to find the position of this rod, when it is ſuſtained in equilibrio by means of a known weight G, appended to the end of the ſtring RFG.

SOLUTION.

Let RCB be the poſition required, and LPO a horizontal line drawn at any diſtance from the point C; to this line, from the points B, C, F, and R, draw the perpendiculars BL, CH, FP and RO, draw BN parallel to LO, cutting the perpendiculars in D, E, and N. From F draw FS perpendicular to FEP, meeting NR, continued in S; and put $BC = m$, $CR = n$, $FP = r$, $DE = HP = p$, $BR = d$, $CH = a$, l for the length of the ſtring RFG, x for the ſine of the angle RBN to the radius, 1; then will $NR = dx$, $BN = d\sqrt{1-xx}$, $DC = mx$, $BD = m\sqrt{1-xx}$, $BE = m\sqrt{1-xx} + p$, $FS = EN = n\sqrt{1-xx} - p$. $DH = BL = a - mx$, $FE = r - a + mx$, $RS = r - a - nx$, or $b - nx$ (putting $r - a = b$) whence $FR = \sqrt{b-nx^2 + n\sqrt{1-xx} - p^2}$ and $PG = r + l - \sqrt{b-nx^2 + n\sqrt{1-xx} - p^2}$; therefore, by Lemma 1, the perpendicular diſtance of the common center of gravity of the two weights B, G, from the horizontal line LO, will be expreſſed by $\frac{rG + lG - G\sqrt{b-nx^2 + n\sqrt{1-xx} - p^2} + aB - mBx}{G+B}$, the fluxion of which (becauſe a maximum) being made equal to nothing, and the equation thence ariſing properly reduced, will give $G \times CR \times RS + G \times FS \times CR \times \tan. \angle RBN = FR \times BC \times B$.



COROL. 1. From C let fall CM perpendicular upon RF (produced if need be) then will $B \times BD = G \times CM$. For by the general equation, we have $\frac{G \times CR \times RS}{FR} + \frac{CR \times FS \times G}{FR} \times \text{tang.} \angle RBN = BC \times B$. Now put s and c for the ſine and coſine of $\angle RBN$ reſpectively, radius being unity, then we ſhall have $\text{tang.} \angle RBN = \frac{s}{c}$, $\text{ſin} \angle FRS = \frac{FS}{FR}$ its coſine $\frac{RS}{FR}$, conſequently $\frac{s \times FS}{FR} + \frac{C \times RS}{FR} = \text{ſin} \angle BRM$; whence

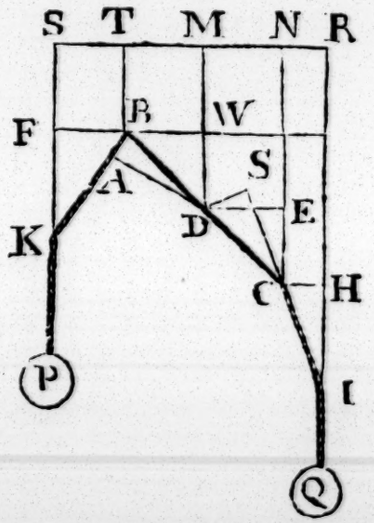
$MC = \frac{s \times FS \times CR}{FR} + \frac{c \times RS \times CR}{FR}$, and $BD = c \times BC$. Now ſubſtitute $\frac{s}{c}$ for the tangent of $<$ RBN in the above general equation, and we ſhall have $\frac{G \times CR \times RS}{FR} + \frac{s \times CR \times FS \times G}{c \times FR} = BC \times B$; multiply each ſide of this equation by c, it becomes $\frac{G \times CR \times RS \times c}{FR} + \frac{s \times CR \times FS \times G}{FR} = BC \times B \times c$, that is, $MC \times G = BD \times B$.

COROL. 2. If the point R falls in the line FP, then $FS = o$, and our equation becomes $RS \times G \times CR = B \times FR \times BC$; but in this caſe RS and FR coincide, therefore when the pulley at the point F has no effect upon the ſtring RFG, the above equation becomes $G \times CR = B \times BC$, a well known property of the common balance.

PROBLEM XXVI.

SUPPOSE BDC an inflexible rod, or lever of the firſt kind, moveable about the fixed point D as a fulcrum; to the extremities B and C thereof, let the ſtrings BKP, CIQ (which, by paſſing freely over the points K and I, ſuſtain the weights P, Q) be faſtened. It is required to find the poſition of the ſaid weights and rod, when they are ſuſtained in equilibrio.

SOLUTION.

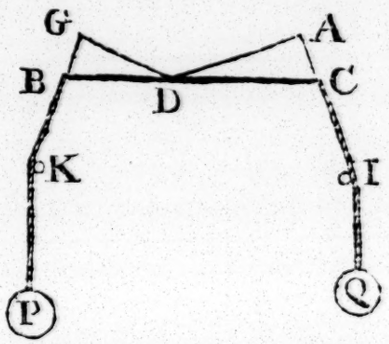


Let BC be the place of the rod or vectis, when ſuſtained at reſt by the weights P, Q. Through any point, as M, draw a horizontal line SMNR; draw DM, BT, CN, perpendiculars to that line; produce PK and QI to S and R. Through the point B, draw FBW parallel to SR; and from the point D, draw DE and CH parallel to SMR. Put the known or given quantities $KS = a$, $DM = b$, $IR = c$, $SM = FW = d$, $MR = e$, $BD = m$, $DC = n$, L and l for the reſpective lengths of the ſtrings BKP, CIQ, and the unknown quantity $ST = FB = x$; then will $BW = d - x$,

$WD = \sqrt{m^2 - d - x^2}$, $TB = FS = s - \sqrt{m^2 - d - x^2}$, and $FK = p + \sqrt{m^2 - d - x^2}$, ($p$ being put for $a - s$). The triangles BWD, DEC, are similar, whence $EC = \frac{n}{m} \times \sqrt{m^2 - d - x^2}$, and consequently $NC = b + \frac{n}{m} \times \sqrt{m^2 - d - x^2}$; put $c - b = q$, then $HI = q - \frac{n}{m} \times \sqrt{m^2 - d - x^2}$; again, $MN = \frac{n}{m} \times \overline{d - x}$, and therefore $e - \frac{n}{m} \times \overline{d - x} = NR$, whence we get $RQ = c + L - \sqrt{q - \frac{n}{m}\sqrt{m^2 - d - x^2}^2 + e - \frac{n}{m}\overline{d - x}^2}$, and $PS = l - \sqrt{p + \sqrt{m^2 - a - x^2}^2 + x^2} + a$; consequently the perpendicular distance of the common center of gravity of the two weights P, Q, from the horizontal line SMR, will (by Lemma 1.) be expressed by $\frac{cQ + LQ - Q\sqrt{q - \frac{n}{m}\sqrt{m^2 - d - x^2} + e - \frac{n}{m}\overline{d - x}^2}}{P + Q}$ $+ aP + Pl - P\sqrt{p + \sqrt{m^2 - \ldots - x^2} + x^2}$ / $P + Q$; which being put into fluxions, and properly reduced, gives $Q \times DC \times HI \times BW \times BK - Q \times DW \times NR \times BK \times DC = P \times BD \times CI \times FK \times BW + P \times BD \times DW \times ST \times CI$.

COROL. 1. From D let fall DA perpendicular upon BK, also DS upon IC (produced if need be) then will $DA \times P = Q \times DS$. By plane trigonometry, we have $\frac{FK}{BK} = \sin \sphericalangle FBK$, and $\frac{FB}{BK}$ its cosine (radius being unity) also $\frac{DW}{BD}$ and $\frac{BW}{BD}$ the sine and cosine of $\sphericalangle DBW$; therefore $\frac{FK \times BW + FB \times DW}{BK \times BD}$ is the sine of $\sphericalangle ABD$, whence $DA = \frac{FK \times BW + FB \times DW}{BK}$. Again, $\frac{CH}{CI}$ and $\frac{HI}{CI}$ are respectively the sine and cosine of $\sphericalangle CIH$, whence the sine and cosine of $\sphericalangle SCE$ will be expressed by $\frac{CH}{CI}$ and $\frac{HI}{CI}$, those of $\sphericalangle DCE$ by $\frac{BW}{BD}$ and $\frac{DW}{BD}$ respectively; whence $\frac{BW}{BD} \times \frac{HI}{CI} - \frac{DW}{BD} \times \frac{CH}{CI} = \sin \sphericalangle DCS$, and consequently $DS = \frac{DC \times BW \times HI - DC \times DW \times CH}{BD \times CI}$. Now it is very evident, from the general equation, that $\frac{Q \times DC \times BW \times HI - Q \times DC \times DW \times CH}{BD \times CI} = \frac{P \times FK \times BW + P \times FB \times DW}{BK}$, therefore $DA \times P = Q \times DS$.

COROL. 2. If $DW = 0$, or the rod BDC be retained by the weights P and Q in an horizontal poſition, then our equation becomes $Q \times DC \times HI \times BK = P \times BD \times CI \times FK$, or, which is the ſame thing, $Q \times DC \times \frac{HI}{CI} = P \times BD \times \frac{FK}{BK}$. But, radius being unity, $\frac{HI}{CI}$ and $\frac{FK}{BK}$ become the ſines of the angles which the directions of the weights make with the rod when at reſt in a poſition parallel to the horizon; and therefore it follows, That if a lever of the firſt kind, as BDC, whoſe fixed point or center of motion is D, be ſuſtained in equilibrio (in a poſition parallel to the horizon) by means of two forces P and Q, acting in any given directions BK, CI, thoſe forces will be in the compounded (inverſe) ratio of the ſines of the angles which the directions of the forces make with the lever, and the reſpective diſtance of each end thereof from the center of motion; that is, $Q : P :: BD \times \text{ſin} \angle KBC : DC \times \text{ſin} \angle BCI$.

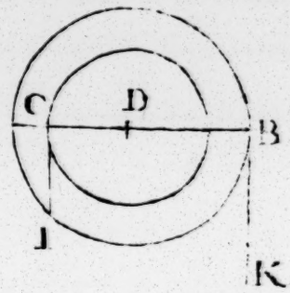


COROL. 3. If the weights P, Q, are freely ſuſpended from the points B, C, of the lever, then we ſhall have $Q : P :: BD : DC$; whence it appears, that in this caſe if the weights Q and P keep the lever at reſt, when in a poſition parallel to the horizon, they will alſo be a juſt counterpoiſe to each other, when the lever is in any other ſituation either above or below the horizontal line, paſſing through D, which is now the place of the common center of gravity of the weights.

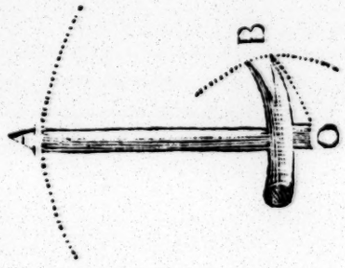
COROL. 4. If from the center of motion D, we draw DG and DA perpendiculars to KB, and IC produced (if neceſſary) then $P : Q :: DA : DG$, for (the lever being ſtill ſuppoſed in a poſition parallel to the horizon) by the firſt Corol. $P : Q :: DC \times \text{ſin} \angle BCI : BD \times \text{ſin} \angle KCB$; and, by plane trigonometry, we have radius (1) : $BD :: \text{ſin} \angle KBC : DG$, whence $DG = BD \times \text{ſin} \angle KBC$; in the ſame manner, $DA = DC \times \text{ſin} \angle BCI$, wherefore $P : Q :: DA : DG$.

From the explication of the lever or vectis, we may, without any farther trouble, deduce the property of the wheel or the axis in peritrochio. For if DC be the radius of the axis, to which a weight is applied in C; and DB the radius of the wheel, or outer circle itſelf, to which a power is applied in B; the line CB, then may be conſidered as a lever or vectis, whoſe fulcrum is the point D, and the power balancing the weight;

weight; that is, the machine being held perfectly ſteady and immovable, the proportion of the power to the weight will be always as the diſtances of the lines of direction from the fulcrum D, reciprocally; that is as D B, C D, reciprocally, or, which is all one, thoſe diſtances being the radii, they will be as the peripheries of the wheel and the axis reciprocally.



From theſe principles are alſo to be deduced the powers of thoſe common machines, Sciſſars, Pincers, Sheers, Forceps, the Crane, the Gimblet, the Hammer (as uſed and applied to the drawing out of a nail) with various others; as for the latter machines, they are more immediately reducible to the axis in peritrochio, though ultimately reſolved into the vectis. The power of the hammer is perhaps equally explicable either way. Thus the power being applied in A the extremity of the handle, and the obſtacle to be removed or drawn out in B, then the end O is the fulcrum or center of motion; and the power moving in a circumference, whoſe radius is OA, and the obſtacle (when moved) in another, whoſe radius is OB. Hence if the forces (of the power, and of the reſiſtance of the obſtacle) are reciprocally as thoſe peripheries, or their reſpective radii (which are the diſtances from the fulcrum) they will ſuſtain one another, and be in equilibrio. This is evident by the preceding corollaries, or by a well known property of the bended lever, which we have inveſtigated at *page* 40.



The explication of the Wedge and the Screw, though they are not ſo immediately and directly explained by the corollaries to the foregoing problem; yet it may not be improper here to give ſome account of them, eſpecially as we have at the beginning of this treatiſe explained ſome properties of the inclined plane; to which the explication of the wedge may be referred, and indeed that of the ſcrew, which is no other than a wedge impelled by a vectis, may alſo be referred. For the wedge, that when the impelling force is to the reſiſtance of the obſtacle, as the thickneſs of the wedge to the height of the ſame; and for the ſcrew, that when the force that turns the ſcrew is to the reſiſtance of the obſtacle, as the interval of the two immediately ſucceeding turnings of the ſpiral to the circumference deſcribed by the power in one converſion of the ſcrew: when it is ſo, the powers (in theſe machines) will be equivalent to the reſiſtances of the obſtacles, or there will be an equilibrium of forces and reſiſtances.

Or, according to Sir *Iſaac Newton*, when the force by which the wedge urges the two ſides of the cleft bodies, is to the force of the mallet upon the wedge as the progreſs of the wedge, according to the determination of the force impreſſed upon it, to the velocity with which the parts of the body give way to the wedge in lines perpendicular to the ſides of the wedge; or when the force of the ſcrew to preſs the body, is to the force of the hand that turns it round, as the circular velocity of the handle to the progreſſive velocity of the ſcrew towards the compreſſed body; then there will be an equilibrium, or the contrary forces and reſiſtances will ſuſtain one another.

The power and uſe of machines conſiſts only in this, that, by diminiſhing the velocity, we may augment the force, and the contrary. From whence, in all ſorts of proper machines, we have the ſolution of this problem, *To move a given weight with a given power*; or, with a given force to overcome any other given reſiſtance. For if machines are ſo contrived, that the velocities of the agent and reſiſtant are reciprocally as their forces, the agent will juſt ſuſtain the reſiſtant; but with a greater diſparity of velocity will overcome it. So that if the diſparity of velocities is ſo great, as to overcome all that reſiſtance which commonly ariſes from the attrition of contiguous bodies, as they ſlide by one another; or from the coheſion of continuous bodies that are to be ſeparated; or from the weights of bodies to be raiſed; the exceſs of the force remaining, after all thoſe reſiſtances are overcome, will produce an acceleration of motion proportional thereto, as well in the parts of the machine as in the reſiſting body. This part of the buſineſs of mechanics may therefore be eaſily compriſed in one general analogy thus: Let p, r, expreſs any powers, moving forces, or forces and reſiſtances, let the velocity of p be expreſſed by m, and that of r by n; then if $p : r :: n : m$, the contrary forces will ſuſtain one another, becauſe (upon this ſuppoſition) $p \times m = r \times n$. Therefore if $p \times m$ be greater than $r \times n$; that is, if $\frac{r}{p}$ be greater than $\frac{m}{n}$, then the force p ſhall overcome the force or reſiſtance r.

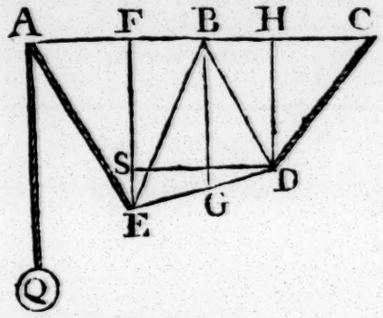
PROBLEM XXVII.

LET ED repreſent a beam ſuſtained at its extremity D, by a ſtring DC of a given length, whoſe end C is faſtened in the horizontal line AC; the end E of the beam is ſupported by means of a weight Q, appended to a ſtring, as EAQ paſſing freely over a ſmall pulley at A. It is

required to determine the poſition of the beam E D, and weight Q, when they are in equilibrio.

SOLUTION.

Let E F and D H be perpendicular to A C, and G be the place of the center of gravity of E D; draw G B parallel to E F, or D H and D S parallel to A C. Put A C $= n$, G D $= a$, G E $= b$, E D $= s$, D C $= l$, E A Q $=$ L, A F $= y$, H C $= x$, and W for the weight of the beam E D; then $\sqrt{l^2-x^2} =$ D H, $n-x-y =$ F H $=$ S D, and $\sqrt{s^2-n-x-y^2}$ $=$ E S, whence $\sqrt{l^2-x^2}+\sqrt{s^2-n-x-y^2} =$ F E. $\frac{a}{s}\sqrt{s^2-n-x-y^2}+\sqrt{l^2-x^2} =$ B G, and



$AE = \sqrt{l^2-x^2+s^2-n-x-y^2+2\sqrt{l^2-x^2}\times\sqrt{s^2-n-x-y^2+y^2}}$; conſequently the perpendicular diſtance of the common center of gravity of the two weights Q and W from the horizontal line A C, will be expreſſed by

$$\frac{LQ-Q\sqrt{l^2-x^2+s^2-n-x-y^2+2\sqrt{l^2-x^2}\times\sqrt{s^2-n-x-y^2+y^2}}}{Q+W}$$

$$+\frac{W\times\frac{a}{s}\times\sqrt{s^2-n-x-y^2}+W\times\sqrt{l^2-x^2}}{Q+W},$$ which muſt be the greateſt poſſible. In fluxions with y conſtant gives (when properly reduced) Q into $\frac{FH-HC}{AE}-\frac{HC\times ES}{DH\times AE}+\frac{FH\times DH}{ES\times AE} =$ W into $\frac{GD\times FH}{ED\times ES}-\frac{HC}{DH}$; or, which is the ſame thing, Q into $\frac{FH\times DH\times FE-FE\times HC\times ES}{AE}$ $=$ W into $\frac{GD\times FH\times DH-HC\times ED\times ES}{ED}$; and by putting the ſame expreſſion into fluxions with x conſtant, we ſhall obtain the following equation, *viz.* $\frac{AH\times ES+FH\times DH}{AE}\times Q=\frac{GD\times FH}{ED}\times W$; divide each ſide of this equation by E D, we have $\frac{AH\times ES+FH\times DH}{AE\times ED}\times Q=\frac{GD\times FH}{ED^2}$ $\times$ W. But theſe factors of Q and W $\times\frac{GD}{ED}$ are reſpectively equal to the ſines of the angles AED, FED, radius being unity. That $\frac{EH}{ED}$ is the ſine

of FED is obvious, and the ſine of AED may be found thus; $\frac{AF}{AE}$ is the ſine of AEF, $\frac{FE}{AE}$ its coſine, $\frac{FH}{ED}$ the ſine of SED, $\frac{ES}{ED}$ its coſine; therefore $\frac{AF \times ES + FH \times FE}{AE \times ED}$ is the ſine of AED. For ES put FE — HD, and the numerator of the fraction becomes $AH \times EF - AF \times HD$; in which for EF write its equal HD + ES, and we get $AH \times HD + AH \times ES - AF \times HD$, or $FH \times HD + AH \times ES$; divide this by $AE \times ED$, and the quotient evidently becomes the ſine of AED, therefore W : Q : : ſin ∢ AED : ſin ∢ FED $\times \frac{DG}{ED}$; which evinces the truth of that principle in mechanics, viz. *That if two forces acting in different directions, as* EF *and* EA, *ſo as to have the ſame effect in ſuſtaining* DE, *or cauſing it to move about a fixed point* D *as a center, muſt be to each other inverſely as the ſines of the angles of incidence* FED, AED. Now imagine the points A, C, to coincide in B, then will AF become FB, HC will become HB; and DC, AE, will reſpectively become BD and BE; whence the foregoing general equations become $\frac{FH \times DH \times FE - FE \times BH \times ES}{BE} \times Q = \frac{GD \times FH \times DH - BH \times ED \times ES}{ED} \times W$, and $\frac{BH \times ES + FH \times DH}{BE} \times Q = \frac{GD \times FH}{ED} \times W$; divide the former of theſe equations by the latter, we have $\frac{FH \times DH \times FE - FE \times BH \times ES}{BH \times ES + FH \times DH} = \frac{GD \times FH \times DH - BH \times ED \times ES}{GD \times FH}$; which, when ES = o, or the beam to be ſupported in a direction parallel to the horizon, becomes FE = DH, and conſequently either of the general equations in this circumſtance $\frac{DH}{BE} \times Q = \frac{GD}{ED} \times W$, or $\frac{\text{ſin} \angle BED}{\text{ſin} \angle FED} \times Q = \frac{GD}{ED} \times W$.

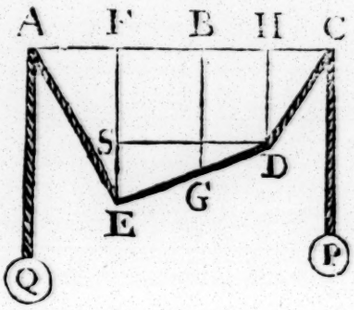
PROBLEM

PROBLEM XXVIII.

LET the beam ED, whoſe center of gravity is at G, be ſuſtained by the weights Q and P, appended to the ends of the ſtrings EAQ, DCP, faſtened to the beam at the points E, D, and paſſing freely over two ſmall pullies fixed at the points A and C in the horizontal line AC. It is required to find the poſition of the ſaid beam and weights, when ſuſtained in equilibrio.

SOLUTION.

Suppoſe it done, and that QAEDCP is the required poſition of the beam and weights. Draw GB, DH and EF, each perpendicular to AC, and DS parallel to it. Put $AC = n$, $GD = a$, $GE = b$, $ED = s$, and W for the weight of the beam ED, L and l for the lengths of the ſtrings EAQ, DCP, reſpectively; let $y = AF$, $x = HC$, $z = AE$; then $\sqrt{zz-yy} = EF$, $n-x-y = FH$, and $\sqrt{ss-\overline{n-x-y}^2} = ES$; hence $\sqrt{zz-yy} - \sqrt{ss-\overline{n-x-y}^2} = FS = HD$,



and $\sqrt{z^2-y^2-2\sqrt{z^2-y^2}\times\sqrt{s^2-\overline{n-x-y}^2}+s^2-\overline{n-x-y}^2+x^2} = DC$, alſo $\frac{a}{s}\times\sqrt{s^2-\overline{n-x-y}^2}+\sqrt{z^2-y^2}-\sqrt{s^2-\overline{n-x-y}^2} = BG$; conſequently we have $\frac{LQ-zQ-\frac{b}{s}\sqrt{s^2-\overline{n-x-y}^2}\times W+\sqrt{z^2-y^2}\times W+}{Q+W+P}$

$\frac{l\times P-P\times\sqrt{z^2-y^2-2\sqrt{z^2-y^2}\times\sqrt{s^2-\overline{n-x-y}^2}+s^2+x^2-\overline{n-x-y}}}{Q+W+P}$

for the perpendicular diſtance of the common center of gravity of the three weights Q, W, P, from the horizontal line AC (by Lemma 1.) this diſtance muſt be the greateſt poſſible; in fluxions, with x and y conſtant, gives $-Q\dot{z}+\frac{z\dot{z}W}{\sqrt{z^2-y^2}} = z\dot{z}-\frac{z\dot{z}}{\sqrt{z^2-y^2}}\times\sqrt{s^2-\overline{n-x-y}^2}\times\frac{P}{DC}$, or $-Q+\frac{AE}{FE}\times W = AE-\frac{AE}{FE}\times ES\times\frac{P}{DC}$; which contracted becomes $\frac{AE}{FE}\times W-\frac{HD\times AE}{FE\times DC}\times P = Q$. Now let the foregoing expreſſion be put into fluxions with z and y conſtant, we ſhall then have

Q

$\frac{EG}{ED} \times FH \times DC \times W = HD \times FH \times P - HC \times ES \times P$; and by taking again the fluxion of the ſaid expreſſion with only y variable, there will ariſe $\frac{AF \times DC \times W}{EF} + \frac{EG \times FH \times DC \times W}{ED \times ES} = \frac{AF \times HD \times P}{EF} + \frac{HD \times FH \times P}{ES}$; which, brought out of the fractions, gives $EG \times FH \times EF \times DC \times W + ED \times ES \times AF \times DC \times W = AF \times DH \times ES \times ED \times P + EF \times HD \times FH \times ED \times P$.

The firſt and ſecond of theſe three general equations may be otherwiſe derived by the application of the common method of dividing forces, as ſhall be ſhewn preſently; but the other, which is alſo abſolutely neceſſary for limiting the poſition of the beam and weights, does not ſeem determinable by any principle of mechanics (except the above) hitherto made uſe of.

Let the force Q, acting in the direction EA, be divided into two other forces, acting in the directions EF, AF; hence AE : EF :: Q : $Q \times \frac{EF}{AE}$ the force in the direction EF; by the ſame way of reaſoning, if we divide the force P, acting in the direction DC, into two other forces acting in the directions DH, CH, we ſhall have $P \times \frac{HD}{DC}$ for the force acting in the direction DH; and conſequently, if we imagine the beam ſupported by thoſe forces, and in thoſe directions, it is evident that $W = P \times \frac{HD}{DC} + Q \times \frac{EF}{AE}$, whence $Q = \frac{AE}{FE} \times W - \frac{HD \times AE}{FE \times DC} \times P$, which exactly agrees with the firſt general equation above-mentioned. The force P is to the weight W of the beam, as the ſine of the angle $HDE \times \frac{EG}{ED}$ to the ſine of the angle CDE, by Problem X; therefore $P \times \text{ſin} \sphericalangle CDE = W \times \text{ſin} \sphericalangle HDE \times \frac{EG}{ED}$; and this equation we are to prove to be the ſame with the ſecond general equation, $\frac{EG}{DE} \times FH \times DC \times W = HD \times FH \times P - HC \times ES \times P$. The ſine of the angle HDE (to radius 1) is $\frac{FH}{ED}$, its coſine $-\frac{ES}{ED}$; the ſine of the angle CDH is $\frac{HC}{DC}$, its coſine $\frac{HD}{DC}$;

therefore the ſine of the angle CDE is $\frac{FH\times DH-ES\times HC}{ED\times DC}$, conſequently $P\times$ ſin $\prec$ CDE is equal to $P\times\frac{FH\times DH-ES\times HC}{ED\times DC}$, and $W\times$ ſin $\prec$ HDE $\times\frac{EG}{ED}$ is equal to $W\times\frac{FH}{ED}\times\frac{EG}{ED}$, whence $P\times\frac{FH\times DH-ES\times HC}{ED\times DC}$ $=W\times\frac{FH\times EG}{ED\times ED}$, or $P\times FH\times DH-HC\times ES\times P=W\times\frac{FH\times DC\times EG}{ED}$, which is the ſame with the ſecond general equation as above. But now to determine the relation of Q to P, by help of theſe equations, let the value of W, in terms of P and other quantities, *viz.* $\frac{DE\times HD\times FH\times P-DE\times HC\times ES\times P}{EG\times FH\times DC}$ be ſubſtituted in the firſt general equation, and we ſhall have $\frac{AE\times HD\times FH\times GD\times P-AE\times DE\times HC\times ES\times P}{FE\times EG\times FH\times DC}$ $=Q$. If for W in the firſt general equation, we ſubſtitute its value, *viz.* $\frac{AF\times HD\times ES\times ED\times P+EF\times HD\times FH\times ED\times P}{FE\times EG\times FH\times EF\times DC+EF\times ED\times ES\times AF\times DC}$, found by the third general equation, we ſhall have (when properly reduced) this equation, $G\times HD\times FH\times AE\times P=EG\times FH\times EF\times DC\times Q+ED\times ES\times AF\times DC\times Q$. But $GD\times HD\times FH\times AE\times P=EG\times FH\times EF\times DC\times Q+ED\times ES\times AE\times HC\times P$, by the laſt equation, expreſſing the relation of Q to P; hence by ſubtraction, we get $AF\times DC\times Q=AE\times HC\times P$ $\therefore Q:P::\frac{AE}{DC}:\frac{AF}{HC}$.

Let us now ſuppoſe HC $=$ o, or the beam ED ſupported in ſuch poſition, that the angle ACD may be a right one; it is then evident, that HD will coincide with DC, and conſequently that the firſt and ſecond general equation (which were alſo determined by the method of dividing forces) will reſpectively become $\frac{AE}{FE}\times W-\frac{AE}{FE}\times P=Q$, and $\frac{EG}{ED}\times W=P$; ſubſtitute for P, in the firſt of theſe equations, its value, *viz.* $\frac{EG}{ED}\times W$, we ſhall have $\frac{AE}{FE}\times W-\frac{AE\times EG}{FE\times ED}\times P=Q$; which, as it involves no apparent abſurdity, may poſſibly be very true, and therefore it does not from hence appear, that the beam cannot be ſuſtained in the propoſed ſituation laſt mentioned in equilibrio, by means of the weights Q and P. Let

now the third general equation, which in this laſt circumſtance becomes $EG \times FH \times EF \times W + ED \times ES \times AF \times W = AF \times ES \times ED \times P + EF \times FH \times ED \times P$, be reſumed, and we have $EG \times FH \times EF \times W + ED \times ES \times AF \times \frac{FE}{AE} \times Q = EF \times FH \times ED \times \frac{EG}{ED} \times W$, conſequently $EG \times FH \times EF \times W - EF \times FH \times EG \times W = ED \times ES \times AF \times \frac{FE}{AE} \times Q$; which being manifeſtly impoſſible, it follows, that the beam cannot be ſuſtained in equilibrio by any two weights Q and P, under the above-mentioned reſtriction with regard to the angle ACD.

COROL. 1. If AF and HC are each ſuppoſed equal to nothing, then muſt EF coincide with AF, and HD with DC; in this caſe, the firſt general equation becomes $W - P = Q$, either of the other two becomes $EG \times W = ED \times P$. For W in this equation ſubſtitute $P + Q$, we have $EG \times Q = GD \times P$; and if we farther ſuppoſe $EG = GD$, then Q muſt be equal to P.

COROL. 2. If the beam ED is to be ſuſtained in a poſition parallel to the horizontal line AC, then $ES = o$, and the firſt equation will become $\frac{AE}{FE} \times W - \frac{AE}{DC} \times P = Q$, either of the other two will be $EG \times DC \times W = HD \times ED \times P$, or, which is the ſame thing, $\frac{EG}{ED} \times W = \frac{EF}{DC} \times P \therefore W = \frac{EF \times ED}{EG \times DC} \times P$; conſequently, if for W in the equation, $\frac{AE}{FE} \times W - \frac{AE}{DC} \times P = Q$, we put its value, we ſhall have $\frac{AE \times ED}{EG \times DC} \times P - \frac{AE}{DC} \times P = Q$; this equation, brought out of fractions, gives $AE \times ED - AE \times EG - \frac{AE}{DC} \times P = Q \times DC \times EG$, that is, $GD \times AE \times P = EG \times DC \times Q$, or $\frac{AE}{DC} \times P = \frac{EG}{GD} \times Q$.

COROL. 3. The general relation of Q to P, is that of $\frac{AE}{DC}$ to $\frac{AF}{HC}$, and, by the laſt corollary, $Q : P :: \frac{AE}{DC} : \frac{EG}{GD}$; therefore when the beam is to be ſupported in a poſition parallel to the horizon, $AF : HC :: EG : GD$.

The

The concluſions derived from the application of the general received principles of mechanics, eſpecially with regard to the reſolution of forces, are not always juſt, and ſometimes even abſurd, as will appear by the following method of finding the ratio of the forces Q, P, and the relation of the angles A and C. By the reſolution of forces, DC : DH : : P : $\frac{P \times DH}{DC}$, the force in the direction DH, and AE : EF : : Q : $\frac{Q \times EF}{AE}$, the force in the direction EF, but $\frac{P \times DH}{DC} : \frac{Q \times EF}{AE}$: : EG : DG, hence P $\times$ DH $\times$ DG $\times$ AE = Q $\times$ EF $\times$ EG $\times$ DC. Again, DC : HC : : P : $\frac{P \times HC}{DC}$, the force of P in the direction CH, and AE : AF : : Q : $\frac{Q \times AF}{AE}$, the force of Q in the direction AF, which, to ſuſtain DE in equilibrio, muſt be equal to the force of P in the oppoſite direction CH, that is, $\frac{P \times HC}{DC}$; otherwiſe AC would have a tendency to move along in a direction parallel to the horizon, which is abſurd, therefore AF $\times$ DC $\times$ Q = HC $\times$ AE $\times$ P, or as HC $\times$ AE : AF $\times$ DC : : Q : P. Now the laſt equation being multiplied by Q $\times$ EF $\times$ DC $\times$ EG = P $\times$ DH $\times$ AE $\times$ DG, we ſhall have HC $\times$ EF $\times$ EG = AF $\times$ DH $\times$ DG, or $\frac{EF \times EG}{AF} = \frac{DH \times DG}{HC}$. Therefore EG : GD : : $\frac{HD}{HC} : \frac{EF}{AF}$, and ſo is the tangent of the angle C to the tangent of the angle A.

This ſolution is the ſame in ſubſtance with that given by Mr. *Simpſon*, at *page* 139 of his firſt treatiſe of Fluxions; and as there did not ſeem to be any apparent error in the application of the diviſion of forces, that ſolution was for a conſiderable time looked upon as juſt. The firſt who diſcovered it to be otherwiſe was (as I have been informed by Mr. *Edmund Stone*) that excellent mathematician, the late *Benjamin Robins*, Eſq; who obſerved, That if the line AE (the other line CD remaining ſtill of the ſame length) ſhould be lengthened until ED became perpendicular to the horizon, the angle at C would then be a right one, and the other at A an acute angle; in that circumſtance, the tangent of the angle at C would be infinite, and therefore could not be to the tangent of an acute angle, in the conſtant ratio of EG to GD. Notwithſtanding the above ſolution cannot be univerſally true, yet in that particular caſe, wherein the beam is ſupported in a direction parallel to the horizon, the concluſion coincides exactly with that derived from our method of inveſtigation, the proportion in ei-

ther being EG : GD : : AF : HC, becauſe EF is then equal to DH.

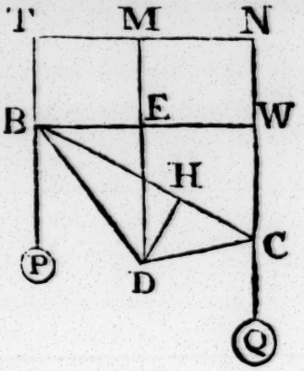
Mr. *Simpſon*, in another ſolution to the ſame problem, ſee *page* 528 of his laſt treatiſe on Fluxions, has indeed avoided the particular error to which it was before liable, by ſhewing that EG : GD : : $\frac{\text{ſin} \measuredangle \text{EDC}}{\text{ſin} \measuredangle \text{CDH}}$: $\frac{\text{ſin} \measuredangle \text{AED}}{\text{ſin} \measuredangle \text{AEF}}$, which (as has been ſhewn at *page* 34 of this treatiſe) is certainly true when the beam is ſupported by two ſtrings of given lengths, faſtened to the horizontal points A, C; but if the beam be ſupported by two weights, P and Q, after the manner deſcribed in our problem, the above concluſion, and the ratio of thoſe weights, as given by Mr. *Simpſon*, will hold no longer true, than while the beam retains a poſition parallel to the horizon, becauſe it is the common center of gravity of the beam and weights that will get into the loweſt place, when they are ſuſtained in equilibrio; and therefore to ſuppoſe, that no alteration in the poſition of the beam would happen, if, inſtead of the weights, the ſtrings were faſtened at thoſe points over which they before were ſuffered to ſlide, is erroneous; for, in this caſe, it is the center of gravity of the beam alone, that muſt be in the loweſt place poſſible, which before was not neceſſary, the equilibrium there depending upon the ſituation of the common center of gravity of the beam and weights; and therefore, unleſs it can be proved, that the perpendicular diſtance of the common center of gravity of the weights and beam (when ſuſtained in equilibrio) from the horizontal line AC, is equal to that of the center of gravity of the beam therefrom, when ſuſtained at reſt, the poſition of the beam in both caſes will not be the ſame.

PROBLEM XXIX.

SUPPOSE BDC a bended lever, moveable about the fixed point D as a fulcrum; from the extremities C and B thereof, let the given weights Q and P be freely ſuſpended, by means of the ſtrings CQ, BP. It is required to find the poſition of this lever BDC, when ſuſtained in equilibrio by the given weights Q, P.

SOLUTION.

Let BDC be the required poſition of the lever. Through any point M, above B, C, draw a horizontal line, as TMN. Draw BT, DM

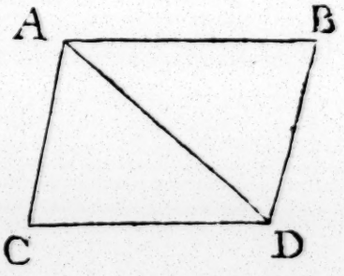


and CN perpendiculars to TMN, and from B, draw BW parallel to TN; join the points B, C, with the right line BC, and thereon from D let fall the perpendicular DH. Put the known or given quantities DM $= b$, BD $= m$, DC $= n$, BC $= h$, fin $<$ DBC $= p$, its cofine $= q$, radius $= 1$, and the unknown quantity BT $= x$; then ED $= b - x$, and conſequently $\sqrt{m^2 - \overline{b-x}^2} =$ BE. Now $m : 1 :: b - x : \frac{b-x}{m} =$ fin $<$ DBE, and $\frac{\sqrt{m^2 - \overline{b-x}^2}}{m}$ its cofine; therefore $q \times \frac{b-x}{m} - p \times \frac{\sqrt{m^2 - \overline{b-x}^2}}{m}$ expreſſes the fine of the angle CBW, whence WC $= qb \times \frac{b-x}{m} - \frac{ph}{m} \times \sqrt{m^2 - \overline{b-x}^2}$, and NC $= x + qb \times \frac{b-x}{m} - \frac{ph}{m} \times \sqrt{m^2 - \overline{b-x}^2}$. Therefore as it is the ſame thing whether the weights are appended at the points Q, P, or C, B, it follows (by Lemma 1.) that the perpendicular diſtance of the common center of gravity of the weights P, Q, from the horizontal line TN, will be expreſſed by $\frac{x\,P + x\,Q + Q\,qb \times \frac{b-x}{m} - \frac{Q\,ph}{m} \times \sqrt{m^2 - \overline{b-x}^2}}{Q + P}$, which muſt be a maximum, and therefore its fluxion equal to nothing, that is, $P\dot{x} + Q\dot{x} - Q \times \frac{qb\dot{x}}{m} - \frac{Q\,ph}{m} \times \frac{\overline{b-x} \times \dot{x}}{\sqrt{m^2 - \overline{b-x}^2}} = 0$; but $\frac{qb}{m} = \frac{BH \times BC}{BD^2}$, and $\frac{ph}{m} = \frac{DH \times BC}{BD^2}$, therefore our equation becomes $P + Q - Q \times \frac{BH \times BC}{BD^2} = Q \times \frac{DH \times BC \times DE}{BD^2 \times BE}$.

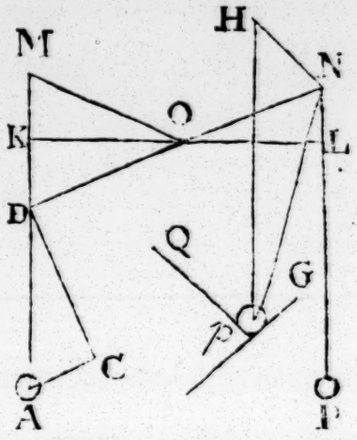
COROL. 1. If BDC be a right angle, then P : Q : : EW : BE; for $BD^2 =$ BC $\times$ BH, and BD : DC : : DE : EW, from this proportion we get BD $\times$ DC $= \frac{BD^2 \times EW}{DE}$. Now for BH $\times$ BC in the above equation, ſubſtitute its equal BD^2, and for DH $\times$ BC, to which BD $\times$ DC is equal, ſubſtitute therein $\frac{BD^2 \times EW}{DE}$, and we have $P = \frac{Q \times EW}{BE}$, conſequently P : Q : : EW : BE.

COROL. 2. P : Q : : EW : BE, let the angle BDC be what it will. For the ſine of the angle BDE put s, c for its coſine, r for the ſine of the angle DBC, t its coſine; then, by plane trigonometry, we have $c\,BD = DE$, $r\,BD = DH$, and $t\,BD = BH$, alſo $s\,BD = BE$: let theſe values be ſubſtituted for their reſpective equals in the above equation, $P + Q - Q \times \frac{BH \times BC}{BD^2} = Q \times \frac{DH \times BC \times DE}{BD^2 \times BE}$, and we ſhall have $P + Q - Q \times \frac{t\,BD \times BC}{BD^2} = Q \times \frac{r\,BD \times BC \times c\,BD}{BD^2 \times s\,BD}$, which contracted becomes $P + Q - Q \times \frac{t\,BC}{BD} = Q \times \frac{rc \times BC}{s\,BD}$, that is, $P = Q \times \frac{ts\,BC + rc\,BC - s\,BD}{s\,BD}$. But it is well known, that $ts + rc$ is the coſine of the difference of the angles BDH and BDE, that is the coſine of the angle EDH, or the right ſine of the angle BCW; conſequently $ts\,BC + rc\,BC$ is equal to BW, whence $P = Q \times \frac{BW - s\,BD}{s\,BD}$, for $s\,BD$ put its equal, *viz.* BE, and then $P = \frac{BW - BE}{BE} \times Q$, that is, $P = \frac{EW}{BE} \times Q$, or P : Q : : EW : BE.

Theſe concluſions agree exactly with what Sir *Iſaac Newton* has determined in his Mathematical Principles of Natural Philoſophy; where, in corollary 2d to the 3d general law of motion, is explained the compoſition of any one direct force AD, out of any two oblique forces AB and BD; and, on the contrary, the reſolution of any one direct force AD, into two oblique forces AB and BD; which compoſition and reſolution of forces are abundantly confirmed from mechanics.



As if the unequal radii OM, ON, drawn from the center O of any wheel, ſhould ſuſtain the weights A and P, by the cords MA and NP; and the forces of thoſe weights to move the wheel were required. Through the center O, draw the right line KOL, meeting the cords perpendicularly in K and L; and from the center O with OL, the greater of the diſtances OK and OL, deſcribe a circle, meeting the cord MA in D; and drawing OD, make AC parallel and DC perpendicular thereto. Now, it being indifferent whether the points K, L, D, of the cords, be fixed to the plane of the wheel or not, the weights will have the ſame effect whether they



they are ſuſpended from the points K and L, or from D and L. Let the whole force of the weight A be repreſented by the line A D, and let it be reſolved into the forces A C and C D; of which the force A C, drawing the radius O D directly from the center, will have no effect to move the wheel: but the other force D C, drawing the radius D O perpendicularly, will have the ſame effect as if it drew perpendicularly the radius O L equal to O D; that is, it will have the ſame effect as the weight P, if that weight is to the weight A, as the force D C is to the force D A, that is, (becauſe of the ſimilar triangles ADC, DOK) as OK to OD or OL. Therefore the weights A and P, which are reciprocally as the radii OK and OL, that lie in the ſame right line, will be equipollent, and ſo remain in equilibrio; which is the well known property of the balance, the lever, and the wheel. If either weight is greater than in this ratio, its force to move the wheel will be ſo much the greater.

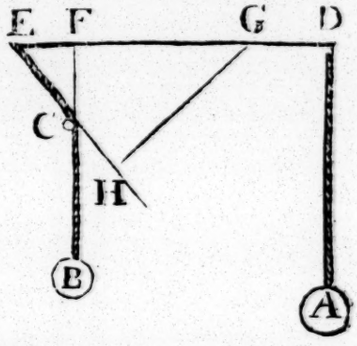
If the weight *p*, equal to the weight P, is partly ſuſpended by the cord N *p*, partly ſuſtained by the oblique plane *p* G, draw *p* H, NH, the former perpendicular to the horizon, the latter to the plane *p* G; and if the force of the weight *p* tending downwards, is repreſented by the line *p* H, it may be reſolved into the forces *p* N, H N. If there was any plane perpendicular to the cord *p* N, cutting the other plane *p* G in a line parallel to the horizon, and the weight *p* was ſupported only by thoſe planes *p* Q, *p* G, it would preſs thoſe planes perpendicularly with the forces *p* N, H N; to wit, the plane *p* Q with the force *p* N, and the plane *p* G with the force H N. And therefore if the plane *p* Q was taken away, ſo that the weight might ſtretch the cord, becauſe the cord, now ſuſtaining the weight, ſupplies the place of the plane that was removed, it will be ſtrained by the ſame force *p* N, which preſſed upon the plane before. Therefore the tenſion of this oblique cord *p* N will be to that of the other perpendicular cord P N, as *p* N to *p* H. And therefore if the weight *p* is to the weight A, in a ratio compounded of the reciprocal ratio of the leaſt diſtances of the cords *p* N, AM, from the center of the wheel, and of the direct ratio of *p* H to P N; the weight will have the ſame effect towards moving the wheel, and will therefore ſuſtain each other.

That the weight *p* is to the weight A, in a ratio compounded of the reciprocal ratio of the leaſt diſtances of the cords *p* N, AM, from the center of the wheel, and of the direct ratio of *p* H to *p* N; that is, that *p* : A :: KO × *p* H : OL × *p* N, may be eaſily ſhewn by help of the IIId Problem,

page 9, where it is proved, that the weight which, acting in the direction pN, is ſufficient to ſuſtain p upon the inclined plane pG, is to the abſolute weight thereof as pH to pN; and therefore if A be taken equal to a fourth proportional to OK, OL and $\frac{p \times p\text{N}}{p\text{H}}$, it is evident, that the weight p, partly ſuſtained by the cord Np, and partly by the plane pG, will be a juſt counterpoiſe to the weight A.

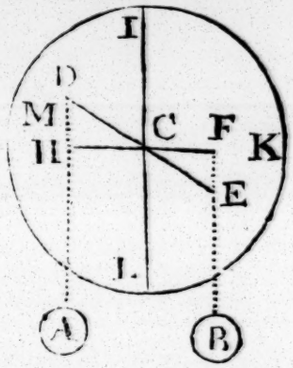
Upon the ſame principles, and in a method ſimilar with Sir *Iſaac Newton*'s, the late Doctor *Helſham*, in his Lectures on Natural Philoſophy, *page* 118, has given the principal properties of the lever in the following manner:

If the weight A hangs freely from one end of a balance, ſo as to have its line of direction DA, perpendicular to the arm of the balance; and if another weight, as B, be hung at the other end E, in ſuch a manner, as that its line of direction EC, by paſſing over a pulley at C, may be oblique to the arm of the balance, the weight B muſt be to the weight A, when it counterbalances it, as EC to CF; that is, as radius to the ſine of the angle CEF, made by the oblique direction of B with the arm of the balance: for if the whole force of gravity in the weight B, acting in the direction EC, be denoted by the line EC, it may be reſolved into two forces, denoted by EF and FC, acting in the directions of thoſe lines; of which two forces, the latter only which acts in the direction FC, perpendicular to the arm of the balance, withſtands the force of gravity in the weight A; the other force, which acts in the direction EF, being intirely employed in preſſing the balance againſt the axis of its motion. Since therefore that part of the weight B, which acts in oppoſition to the weight A, is to the whole weight B, as FC to EC; it is manifeſt, that in order to make the weight B balance the weight A, it muſt exceed the weight A in the ſame proportion that the line EC exceeds the line FC.

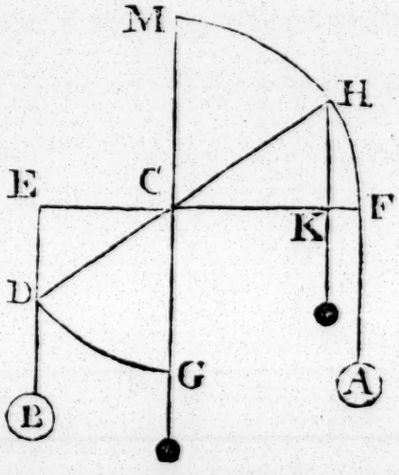


As a corollary it follows, that the perpendicular diſtances of the lines of direction from the center of motion, are to one another inverſely as the weights; for if from G, the center of motion, be let fall GH perpendicular to EC, that line will be the perpendicular diſtance of the direction EC from G; and EG, equal to DG, is the perpendicular diſtance of the direction DA; but the triangles EFC and EHG are ſimilar, becauſe their angles at E are equal, and they have each a right angle;

but the weight B is to the weight A, as EC to FC; that is, as EG or DG to HG: ſo that wherever two powers, which act in oblique directions, are to one another in the inverſe ratio of the perpendicular, or ſhorteſt diſtance of their lines of direction from the center of motion, they muſt balance one another. Whence it follows, that if two weights, as A and B, be ſuſpended from two points, as D and E, in the plane of a wheel placed in a vertical poſition; and if the line DE, which is drawn through the two points of ſuſpenſion, paſſes through C the center of motion, the weights will balance, provided they be to one another inverſely as the diſtances of their points of ſupenſion from the center of motion; that is, if A be to B, as CE to DC: for ſince the weights hang freely, their lines of direction DA and FB will be perpendicular to the horizon, and, of conſequence, parallel to each other; wherefore if the line HCF be drawn through the center of motion, perpendicular to the two lines DA and FB, the triangles DHC and ECF will be ſimilar; conſequently DC will be to EC, as HC to FC; but by ſuppoſition, the weight A is to the weight B, as CE to CD; that is, as CF to CH: ſo that the weights are to one another inverſely, as the perpendicular diſtances of their lines of direction from the center of motion, conſequently they muſt balance; and though the wheel ſhould be turned upon its axis, and the diſtances of the lines DA and EB from C be thereby altered, yet will the ſimilarity of the foremen-tioned triangles continue, and, of conſequence, the balance between the weights will be preſerved.



If a crooked lever FCD be ſo placed on its prop at C, as that the arm CF may be parallel to the plane of the horizon, and the arm CD inclined thereto; if two weights, as B and A, appended at D and F, be in the reciprocal proportion of the perpendicular diſtances of their lines of direction from the prop; that is, if B be to A, as FC to EC, there will be a balance; for as long as the arm CF continues parallel to the horizon, the weight B, hanging from the point D, acts in the ſame manner in oppoſition to the weight A, as if it hung from E, the extremity of the ſtrait lever FC, continued on to E; in which caſe the weight B, that balances the

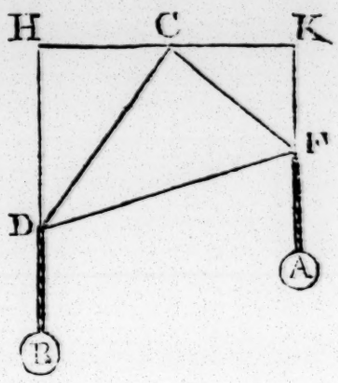


weight A, muſt bear the ſame proportion to it, that FC does to EC. If therefore the arm DC be bent in ſuch a manner, as that EC may be one half or one third of FC; in the former caſe a weight of two ounces, and in the latter one of three ounces, hanging from D, will be counterpoiſed by one ounce hanging from F.

If, by moving the lever, the arm FC be put out of its parallel direction, the balance will be deſtroyed; for that cannot be preſerved, unleſs the diſtance of B's direction from the prop, continues to bear the ſame proportion to the diſtance of A's direction, that EC does to FC; which in this caſe is impoſſible. For, firſt, if the point F be moved upward towards H, and of courſe the point D downward towards G, it is manifeſt, that the diſtances of both directions will be leſſened; but the decreaſe of EC in a given time, will bear a greater proportion to the decreaſe of FC, than EC does to FC; for by that time the point D has moved from D to G, through the arch DG, which meaſures the angle of CD's inclination, EC will vaniſh; whereas FC cannot vaniſh till ſuch time as the point F has moved from F to M, through the quadrantal arch FM; but in the ſame time that the point D moves from D to G, through the arch DG, the point F can move only from F to H, through the arch FH, ſimilar to DG; which arch being always leſs than the quadrant, the perpendicular diſtance of A's direction from the prop, to wit, FC, will not vaniſh upon the arrival of the point F at H; that is, it will not vaniſh ſo ſoon as EC; conſequently the decreaſe of EC in a given time, muſt bear a greater proportion to the decreaſe of FC, than EC does to FC: wherefore EC, as diminiſhed in any given time, will be to FC, as diminiſhed in the ſame time, in a leſs proportion than that of EC to FC; or, in other words, the perpendicular diſtance of B's direction from the prop, will bear a leſs proportion to the perpendicular diſtance of A's direction, than EC does to FC; and therefore the weight A will preponderate. If the point F be moved downward, and conſequently D upward, it is manifeſt, from the inſpection of the figure, that the diſtance of A's direction from the prop continually diminiſhes, at the ſame time that the diſtance of B's direction increaſes; and therefore the weight B muſt in that caſe overbalance the weight A.

In order to examine the foregoing property by our method; let BCF repreſent a crooked lever, whoſe prop is at the point C, and to whoſe ends D and F, the given weights B and A are appended. Through the point C, draw a right line HCK, parallel to the horizon; from D and F, draw DH and FK perpendicular to HK.

Put



Put s and c for the ſine and coſine of the angle DCF, x and y for thoſe of the angle FCK, radius $= 1$, $DC = a$, $CF = b$; then, by plane trigonometry, we have $1 : b :: x : bx = FK$, and $sy + cx = \text{ſin} < DCH$, therefore $DH = asy + acx$, and conſequently (by Lemma 1.) the perpendicular diſtance of the common center of gravity of the weights A, B, from the horizontal line HK, will be expreſſed by $\frac{Basy + Bacx + Abx}{A + B}$, the fluxion of which being made equal to nothing, gives $Bas\dot{y} + Bac\dot{x} + Ab\dot{x} = 0$, or $Bas\dot{x} - Bac\sqrt{1-x^2} = Ab\sqrt{1-xx}$, by ſubſtituting for y its value $\sqrt{1-x^2}$, and dividing each ſide of the equation by $\dot{x}$, that is, $B \times DC \times \text{ſin} < DCF \times \text{tan} < FCK - B \times DC \times \text{coſin} < DCF = A \times CF$.

COROL. 1. $B : A :: CK : HC$; for $CK = b\sqrt{1-x^2}$, and becauſe $sy + cx$ expreſſes the ſine of the angle HCD, $sx - cy$ will expreſs its coſine, therefore $HC = asx - acy$, or $asx - ac\sqrt{1-x^2}$; conſequently, if for $asx - ac\sqrt{1-x^2}$, and $b\sqrt{1-x^2}$ in the above equation, we ſubſtitute their reſpective values juſt now found, that is, HC and CK, we ſhall have $B \times HC = A \times CK$, and therefore $B : A :: CK : HC$.

COROL. 2. If $x = 0$, our equation becomes $-Bac = Ab$; which ſhews, that unleſs c be negative, or the angle DCF be obtuſe, no equilibrium of the weights can enſue, when the arm CF is in a ſituation parallel to the horizon.

Doctor *Helſham*, in his ſolution obſerves, that in order to preſerve an equilibrium between the weights A, B, when the arm CF (ſee the laſt figure but one) is parallel to the horizon, the weight A muſt be to the weight B, in the reciprocal proportion of the perpendicular diſtances of their lines of direction from the prop; but from our inveſtigation it appears, that in whatever ſituation the lever may happen to be placed, there will be an equilibrium between the weights, provided they are in the reciprocal ratio of the perpendicular diſtances of their lines of direction from the prop. It is indeed true, that, if after having once fixed the quantity of each weight, and conſequently their ratio, and found the poſition in which the lever will ſuſtain thoſe weights in equilibrio, it be put out of that poſition, the balance will be deſtroyed; but this does not in any wiſe render our ſolution

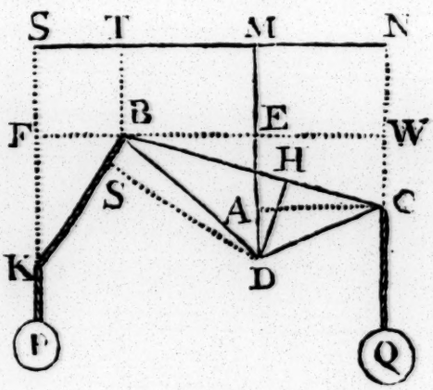
leſs general, as is evident from the corollaries. And therefore what the Doctor has advanced with regard to the paralleliſm of the arm CD of the lever with the horizon, does not appear at all eſſential to the ſolution of his problem; becauſe, as we have proved above, the direct ratio of the weights muſt be the reciprocal ratio of the perpendiculars drawn from the center of motion, to the directions of the weights reſpectively.

PROBLEM XXX.

SUPPOSE BDC a bended lever, moveable about the fixed point D as a fulcrum; at the extremity C thereof, let a given weight Q, by means of a ſtring, be freely ſuſpended; and at the other extremity B of this lever, let a ſtring BKP be faſtened, which, by paſſing over a ſmall pulley fixed in the point K, ſuſtains a given weight P. It is required to find the poſition of the lever BDC, when the weights P and Q are a juſt counterpoiſe to each other.

SOLUTION.

Let BDC be the required poſition of the lever. Through any point M, draw a line parallel to the horizon, as SMN. Draw KS, DM and CN, each perpendicular to SMN, and through B, draw TBW parallel to SN. Join the points B and C with the right line BC; and thereon from D, let fall the perpendicular DH. Put the known or given quantities $KS = a$, $DM = b$, $CN = c$, $SM = d$, $BD = m$, $DC = n$, $BC = b$, L for the length of the ſtring BKP, p and q for the ſine and coſine of the angle DBC, to the radius 1; the unknown quantity $TB = x$. Then $DE = b - x$, $BE = \sqrt{m^2 - \overline{b-x}^2}$, and $FB = d - \sqrt{m^2 - \overline{b-x}^2}$, whence $BK = \sqrt{\overline{d - \sqrt{m^2 - \overline{b-x}^2}}^2 + \overline{a-x}^2}$, and conſequently $PS = L - \sqrt{\overline{d - \sqrt{m^2 - \overline{b-x}^2}}^2 + \overline{a-x}^2} + a$. Again, $\frac{b-s}{m}$ = ſin $<$ DBE, and $\frac{\sqrt{m^2 - \overline{b-x}^2}}{m}$ = its coſine, therefore $q \times \frac{b-x}{m} - \frac{p}{m}$



$\times \sqrt{m^2 - \overline{b-x}^2} = \sin \measuredangle$ WBC, whence NC becomes $qb \times \frac{b-x}{m} - \frac{bp}{m} \times \sqrt{m^2 - \overline{b-x}^2} + x$; therefore, by Lemma 1, the perpendicular distance of the common center of gravity of the weights P, Q, from the horizontal line SMN, will be expreſſed by $\frac{P \times L - P \times \sqrt{\overline{d - \sqrt{m^2 - \overline{b-x}^2}}^2 + \overline{a-x}^2}}{P+Q} + \frac{P \times a + Q \times qb \times \frac{b-x}{m} - Q \times \frac{ph}{m} \times \sqrt{m^2 - \overline{b-x}^2} + x}{P+Q}$, which muſt be the greateſt poſſible. In fluxions we have $P \times \frac{-\overline{d - \sqrt{m^2 - \overline{b-x}^2}} \times \frac{\overline{b-x} \times \dot{x}}{\sqrt{m^2 - \overline{b-x}^2}} - \overline{a-x} \times \dot{x}}{\sqrt{\overline{d - \sqrt{m^2 - \overline{b-x}^2}}^2 + \overline{a-x}^2}} = \dot{x} - \frac{qb\dot{x}}{m} - \frac{bp}{m} \times \frac{\overline{b-x} \times \dot{x}}{\sqrt{m^2 - \overline{b-x}^2}} \times Q$; which, being properly reduced, becomes $P \times \frac{FB \times DE + BE \times FK}{BK} = Q \times \frac{BC \times BE}{BD} \times \cos \measuredangle$ DBC $+ Q \times \frac{BC \times DE}{BD} \times \sin \measuredangle$ DBC $- Q \times$ BE.

COROL. 1. If FB $= o$, or the weight P freely ſuſpended from the extremity B of the arm DB, then the above general equation will contract into $P \times BE = Q \times \frac{BC \times BE}{BD} \times \cos \measuredangle$ DBC $+ Q \times \frac{BC \times DE}{BD} \times \sin \measuredangle$ DBC $-$ BE $\times$ Q.

COROL. 2. If BDC be a right angle, and FB ſtill $= o$, our laſt equation will farther contract into $P \times BE \times BD = Q \times DC \times DE$.

The concluſions in theſe corollaries agree exactly with the determinations in the corollaries to the laſt problem, *page* 63; and therefore it follows, that in either of theſe, P : Q : : EW : BE.

COROL. 3. P is to Q, as EW is to the perpendicular DS, drawn from the fulcrum or center of motion D, to the line of direction BK of the weight or power P, the angle BDC being either obtuſe or acute. In order to ſhew this, draw CA perpendicular to DM, then will CA $=$ WE. Put s and c for the ſine and coſine of the angle BDE, r and t for the ſine and

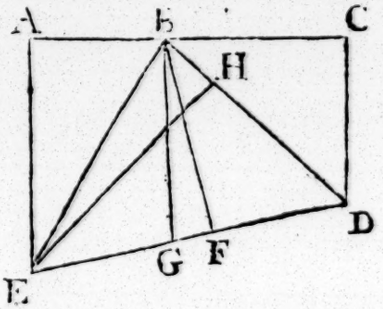
coſine of the angle DBC; then, by plane trigonometry, we have $DE = c\,BD$, $DH = r\,BD$, $BH = t\,BD$, $BE = s\,BD$. Let theſe values of DE, DH, BH, and BF, be ſubſtituted in the general equation above found, it will become $\frac{P \times FB \times DE + BE \times FK \times P}{BE \times BK} = \frac{t \times BD \times BC}{BD^2} \times Q + \frac{r\,BD \times BC \times c\,BD}{BD^2 \times BE} \times Q - Q$; whence, by proper reduction, we get $\frac{P \times FB \times DE + BE \times FK \times P}{BE \times BK} = \frac{-Q \times s\,BD + ts\,BC \times Q}{BE} + \frac{rc\,BC \times Q}{BE}$, therefore $P \times \frac{FB \times DE + BE \times FK}{BD \times BK} = \frac{Q \times EW}{BD^2}$; for $ts + rc$ is the ſine of the angle BCW, and that of the angle SBD is expreſſed by $\frac{FB \times DE + BE \times FK}{BD \times BK}$, conſequently $P \times SD = Q \times EW$, and thence $P : Q :: EW : DS$.

PROBLEM XXXI.

TO determine the poſition of the beam ED, when ſuſpended at reſt by means of the ſtring BDEB (faſtened to the point B in the horizontal line AC) and upon which the ſaid beam freely ſlides.

SOLUTION.

Suppoſe EBD to be the required poſition. From the points E, D, draw EA, DC, perpendiculars to AC; join the point G the center of gravity of the beam, and the point of ſuſpenſion B with the right line BG. From E draw EH perpendicular to DB, and put $EG = m$, $GD = n$, $m + n = d$, $BE + BD = s$, $BD = x$, $\sin \angle DBC = z$, radius $= 1$; then, by Lemma 2, we have $\frac{\overline{s-x}^2 + x^2 - d^2}{2x \times \overline{s-x}} = \cos \angle EBD$, and



therefore its ſine $= \frac{\sqrt{2x \times \overline{s-x}^2 - \overline{s-x}^2 + x^2 - d^2}^2}{2x \times \overline{s-x}}$; conſequently the ſine of the angle EBA will be expreſſed by $\frac{\sqrt{2x \times \overline{s-x}^2 - \overline{s-x}^2 + x^2 - d^2}^2 \times \sqrt{1 - z^2}}{2x \times \overline{s-x}} + z$

$+z\times\frac{\overline{s-x^2}+x^2-d^2}{2x\times\overline{s-x^2}}$, therefore $AE=\frac{\sqrt{2x\times\overline{s-x^2}-\overline{\overline{s-x^2}+x^2-d^2}^2}\times\sqrt{1-z^2}+z\times\overline{\overline{s-x^2}+x^2-d^2}}{2x}$ and $DC=zx$; hence, by Lemma 1, the perpendicular diſtance of the center of gravity of the beam from the horizontal line AC, becomes $\frac{n}{d}\times\frac{\sqrt{2x\times\overline{s-x^2}-\overline{\overline{s-x^2}+x^2-d^2}^2}\times\sqrt{1-z^2}}{2x}+z\times\frac{\overline{\overline{s-x^2}+x^2-d^2}}{2x}-zx\times\frac{n}{d}+zx$, which muſt be the greateſt poſſible; in fluxions, with x conſtant, gives $\frac{n}{d}\times\frac{\sqrt{2x\times\overline{s-x^2}-\overline{\overline{s-x^2}+x^2-d^2}^2}}{2x}\times\frac{z\dot{z}}{\sqrt{1-z^2}}=\frac{n}{d}\times\dot{z}\times\frac{\overline{\overline{s-x^2}+x^2-d^2}}{2x}+\frac{m}{d}\times x\dot{z}$. Divide each ſide of the equation by $\frac{\dot{z}}{d}$, and for n, m, $\frac{\sqrt{2x\times\overline{s-x^2}-\overline{\overline{s-x^2}+x^2-d^2}^2}}{2x}$, $\frac{\overline{\overline{s-x^2}+x^2-d^2}}{2x}$ and x; ſubſtitute their reſpective values, *viz.* GD, EG, EH, BH and BD, there will ariſe $GD\times EH\times\frac{z}{\sqrt{1-z^2}}=DG\times BH+EG\times BD$, whence $\frac{z}{\sqrt{1-zz}}$, or the tangent of the $\sphericalangle$ DBC to the radius, 1, is equal to $\frac{DG\times BH+EG\times BD}{GD\times EH}$. Let the expreſſion for the maximum be put into fluxions with z conſtant, and the equation thence ariſing, will, after proper reduction, become $EG\times BD=GD\times EB$, therefore EG : GD :: EB : BD, and conſequently the right line BG biſects the angle EBD.

The ſame otherwiſe.

It appears, by the general principle of mechanics, mentioned at *page* 33, that when the beam ED is ſuſtained at reſt by means of the ſtring BDEB, the right line BG, joining the center of gravity G of the beam and point of ſuſpenſion B, will be perpendicular to the horizontal line AC. It is likewiſe evident, that the center of gravity of the beam will be the fartheſt poſſible from the point of ſuſpenſion, and conſequently that BG muſt be a maximum: this being premiſed, and other things remaining as before, let BF be drawn perpendicular to ED; then, by plane trigonometry, we have $d:s::s-2x:\frac{s}{d}\times\overline{s-2x}=EF-FD$, hence $\frac{d}{2}+\frac{s}{2d}\times\overline{s-2x}$

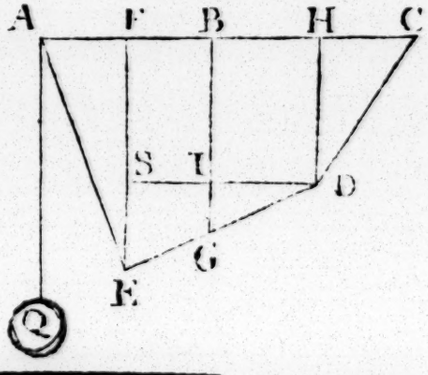
$= EF$, and $BF^2 = \overline{s-x}^2 - \left|\frac{d}{2} + \frac{s}{2d} \times \overline{s-2x}\right|^2$, but $GF^2 = \left|\frac{d}{2} \times \frac{s}{2d} \times \overline{s-2x-m}\right|^2$; therefore $BG^2 = \overline{s-x}^2 - \left|\frac{d}{2} + \frac{s}{2d} \times \overline{s-2x}\right|^2 + - m + \frac{d}{2} + \frac{s}{2d} \times \overline{s-2x}\Big|^2$; the fluxion of which, being made equal to nothing, gives $s - x = \frac{d}{2} + \frac{s}{2d} \times \overline{s-2x} \times \frac{s}{d} + \frac{d}{2} + \frac{s}{2d} \times \overline{s-2x-m} \times \frac{s}{d}$, or, which is the ſame thing, $BE = EF \times \frac{s}{d} - GF \times \frac{s}{d}$, that is, $BE = EG \times \frac{EB + BD}{ED}$; whence $EG \times BD = GD \times EB$, and conſequently $EG : GD :: EB : BD$, the ſame as before.

PROBLEM XXXII.

LET QAEDC be a ſtring of a given length, faſtened at C, and paſſing over a ſmall pulley fixed in the point A of the horizontal line AC; upon this ſtring, ſuppoſe a beam ED, whoſe center of gravity is G, freely ſlides. It is required to determine the poſition thereof, when ſuſtained in equilibrio by means of a known weight Q, appended to the extremity of the given ſtring.

SOLUTION.

From the points E, G, and D, draw EF, GB, and DH, perpendiculars to AC. Draw DS parallel to AC, and put l for the length of the ſtring QAEDC, $AC = n$, $DG = a$, $EG = b$, $ED = s = a + b$; then will $QAE + DC = l - s$, which put $= p$, and let W repreſent the weight of the beam ED; call HC, x, AF, y, and AE, z, then will $FE = \sqrt{z^2 - y^2}$, $ES = \sqrt{s^2 - \overline{n-x-y}^2}$, and $DH = \sqrt{zz-yy} - \sqrt{s^2 - \overline{n-x-y}^2}$, therefore we have $DC = \sqrt{z^2 - y^2 + s^2 - \overline{n-x-y}^2 - 2\sqrt{z^2-y^2} \times \sqrt{s^2 - \overline{n-x-y}^2} + x^2}$. By the ſimilar triangles DES, DGI, we have $DE : ES :: DG : GI = \frac{a}{s}$



$\times \sqrt{s^2 - \overline{n-x-y}^2}$, to which add HD, gives $BG = \sqrt{z^2-y^2} - \frac{b}{s} \times \sqrt{s^2 - \overline{n-x-y}^2}$; whence, by Lemma 1, the perpendicular distance of the common center of gravity of the weight and beam from the horizontal line AC, will be expreſſed by $\frac{pQ-Qz-Q\times\sqrt{z^2-y^2+s^2-\overline{n-x-y}^2-2\sqrt{z^2-y^2}}}{Q+W} \times \sqrt{s^2-\overline{n-x-y}^2+x^2} + W\times\sqrt{z^2-y^2} - W\times\frac{b}{s}\times\sqrt{s^2-\overline{n-x-y}^2}$, which

$Q+W$

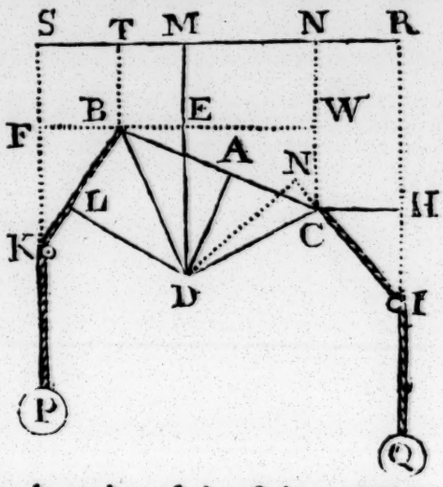
muſt be the greateſt poſſible. In fluxions, with z and y invariable, gives, when properly reduced, $W\times\frac{GE\times FH}{ED} = Q\times\frac{DH\times FH - HC\times ES}{DC}$; the ſame expreſſion being put into fluxions, with only z variable, produces this equation, $Q\times FE\times DC + HD\times AE\times Q = AE\times DC\times W$; and, laſtly, by making y flow, x and z being invariable, we have, after due reduction, $FH\times\frac{GE}{ED}\times W + W\times\frac{AE\times ES}{FE} = Q\times\frac{HD\times FH\times FE + HD\times AE\times ES}{DC\times FE}$.

COROL. If ES = 0, our equations become $W\times\frac{GE}{ED} = Q\times\frac{DH}{DC}$, $AE\times DC\times W = Q\times HD\times DC + Q\times HD\times AE$, and $\frac{FH\times GE}{ED}\times W = Q\times\frac{HD\times FH}{DC}$, or $\frac{GE}{ED}\times W = Q\times\frac{HD}{DC}$. The firſt and third of theſe equations are exactly alike; and therefore if we divide the ſecond equation by either of theſe, we ſhall have $\frac{AE}{GE} = \frac{DC+AE}{ED}$; wherein, if we ſuppoſe $EG = GD$, it will follow that $AE = DC$.

PROBLEM XXXIII.

TO determine the poſition in which the bended lever BDC, whoſe center of motion is the point D, will be ſuſtained in equilibrio by means of two weights P, Q, appended to the ends P and Q of the ſtrings BKP, CIQ, faſtened to the extremities B and C of the lever, and paſſing freely over two ſmall pullies fixed in the points K, I.

SOLUTION.



Through any convenient point M, draw a horizontal line SMR, and let BDC repreſent the poſition required. Produce PK, QI, to the line SMR, and draw BT, DM, CN, parallel to PS or QR; draw alſo BEW and CH parallel to SMR; join the points B, C, with a right line BC, and from D, draw DA perpendicular to BC. Put KS $= a$, DM $= b$, SM $= d$, BD $= m$, DC $= n$, BC $= h$, IR $= c$, MR $= e$, p and q for the ſine and coſine of the angle DBC, radius $= 1$. Let L and l repreſent the lengths of the ſtrings BKP, CIQ, put TB $= x$; then will DE $= b - x$, BE $= \sqrt{m^2 - \overline{b-x}^2}$, and FB $= d - \sqrt{m^2 - \overline{b-x}^2}$, whence BK $= \sqrt{\overline{d - \sqrt{m^2 - \overline{b-x}^2}}^2 + \overline{a-x}^2}$, and conſequently PS $= L - \sqrt{\overline{d - \sqrt{m^2 - \overline{b-x}^2}}^2 + \overline{a-x}^2} + a$. Again, $\frac{b-s}{m}$ $=$ ſin $\measuredangle$ DBE, and $\frac{\sqrt{m^2 - \overline{b-x}^2}}{m}$ $=$ its coſine, therefore $q \times \frac{b-x}{m} - \frac{p}{m} \times \sqrt{m^2 - \overline{b-x}^2} =$ ſin $\measuredangle$ WBC; whence NC $= qb \times \frac{b-x}{m} + x - \frac{ph}{m} \times \sqrt{m^2 - \overline{b-x}^2}$, and HI $= c - qb \times \frac{b-x}{m} + \frac{hp}{m} \times \sqrt{m^2 - \overline{b-x}^2} - x$. By a well known theorem in plane trigonometry, mentioned at *page* 39, we have $p \times \frac{b-s}{m} + q \times \frac{\sqrt{m^2 - \overline{b-x}^2}}{m}$ for the ſine of the angle BCW; and therefore EW $= ph \times \frac{b-s}{m} + qh \times \frac{\sqrt{m^2 - \overline{b-x}^2}}{m} - \sqrt{m^2 - \overline{b-x}^2}$, and CI $=$

$$\sqrt{\left|c - qb \times \tfrac{b-x}{m} + hp \times \tfrac{\sqrt{m^2 - \overline{b-x}^2}}{m} - x\right|^2 + \left|e - ph \times \tfrac{b-s}{m} - qb \times \tfrac{\sqrt{m^2 - \overline{b-x}^2}}{m} + \sqrt{m^2 - \overline{b-x}^2}\right|^2}$$

Hence, by Lemma 1, the perpendicular diſtance of the common center of gravity of the two weights P, Q, from the horizontal line SR, will be expreſſed by

$$\frac{P \times L - P \times \sqrt{\overline{d - \sqrt{m^2 - \overline{b-x}^2}}^2 + \overline{a-x}^2} + P \times a + l \times Q - Q \times \sqrt{\left|c - qb \times \tfrac{b-x}{m} + \tfrac{hp}{m} \times \sqrt{m^2 - \overline{b-x}^2} - x\right|^2 + \left|e - ph \times \tfrac{b-s}{m} - \tfrac{qh}{m} \times \sqrt{m^2 - \overline{b-x}^2} + \sqrt{m^2 - \overline{b-x}^2}\right|^2}}{P + Q}$$

$$+ \frac{Q \times c}{P + Q}$$

; which being put into fluxions, made $= 0$, and the equation

thence

thence ariſing properly reduced, by firſt ſubſtituting therein for the values of $\frac{qh}{m}$ and $\frac{ph}{m}$ their reſpective equals, *viz.* $\frac{BA \times BC}{BD^2}$ and $\frac{AD \times BC}{BD^2}$, there will at length be produced the following equation, $\frac{P \times FB \times DE + P \times BE \times KF}{BE \times BK} + \frac{Q \times HI}{CI} + \frac{Q \times CH \times DE \times AB \times BC}{BD^2 \times BE \times CI} - \frac{Q \times HI \times BA \times BC}{BD^2 \times CI} = \frac{Q \times HI \times AD \times BC \times DE}{BD^2 \times BE \times CI}$.

COROL. 1. If FB and CH are each $= o$, we ſhall have $P + Q = Q \times \frac{BA \times BC}{BD^2} + \frac{Q \times AD \times BC \times DE}{BD^2 \times BE.}$.

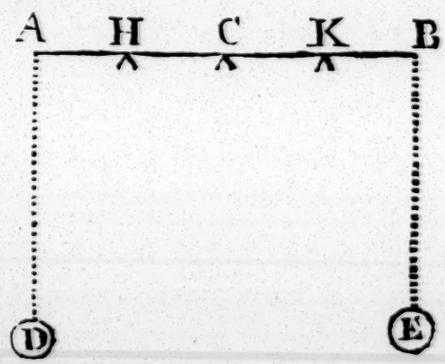
COROL. 2. If FB, CH and AD are each $= o$, the general equation will contract into $AB \times P = AC \times Q$, or $P : Q :: AC : AB$, A in this caſe being the center of motion.

COROL. 3. If, from the center of motion D, we let fall DL perpendicular upon BK, and DN perpendicular upon IC produced, then will $P \times DL = Q \times DN$; whence $P : Q :: DN : DL$.

Before we quit this ſubject of the lever, it may not be improper to take notice of one other property thereof, which is ſomewhat ſingular and ſurpriſing; namely, That if a man, ſtanding in one ſcale of the common balance, and counterpoiſed by a weight in the other, lays his hand to any part of the beam, and preſſes it upwards, he will thereby deſtroy the equilibrium, and make the ſcale wherein he ſtands to preponderate.

Doctor *Helſham*, at *page* 91 of his Lectures in Natural Philoſophy, accounts for this property in the following manner:

Let AB repreſent the beam of a common balance or pair of ſcales, playing on the axis at C; and let a man, ſtanding in the ſcale D, and counterpoiſed by a weight in the ſcale E, lay his hand to ſome part of the beam, either on the ſame ſide of the axis with himſelf, as at H, or on the other ſide as at K, and preſs the ſame upward, inaſmuch as action and reaction are always equal; it is manifeſt, that with

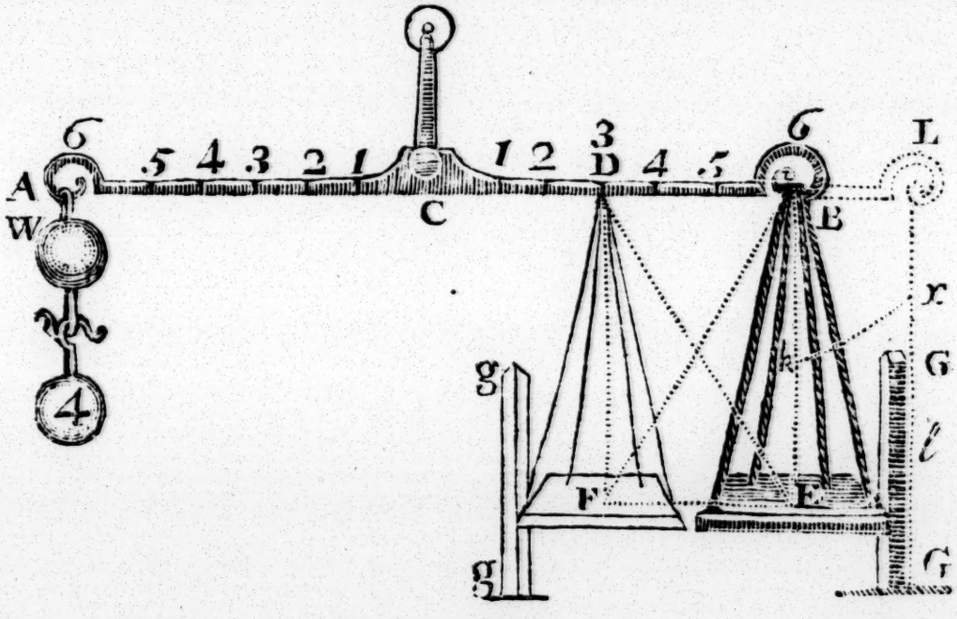


whatever force the hand preſſes upward, againſt the point H or K; with the ſame the hand, and conſequently the man's whole body, is preſſed downward; and therefore the ſcale D wherein he ſtands, bears the ſame preſſure from his feet, that the point H or K does from his hand: but the preſſure upon the ſcale D, may be looked upon as applied to the beam at the point A, from which the ſcale hangs; conſequently, the ſame force which preſſes up the point H or K, preſſes down the point A. Wherefore, putting F to denote that force, $F \times HC$ will expreſs the moment wherewith the arm AC is preſſed upward, when the hand is applied at H; and $F \times KC$ the moment wherewith the arm BC is preſſed upward, the hand being applied at K; and in both caſes, $F \times AC$ will expreſs the moment wherewith the arm AC is preſſed downward by means of the reaction: if therefore the hand be applied at H, it is manifeſt, that as the arm AC is at one and the ſame time preſſed upward by a force, which is as $F \times HC$, and downward by a force which is as the ſame $F \times AC$, and as HC is ever leſs than AC, the arm AC muſt deſcend with the difference of thoſe forces; that is, with a force equal to $F \times AH$, which is the diſtance of the hand from the point A. If the hand be applied at K, the arm CB is preſſed upward, and conſequently AC downward, with a force equal to $F \times KC$; and, upon account of the reaction, AC is likewiſe preſſed downward with a force equal to $F \times AC$, and therefore it muſt deſcend with a force equal to the ſum of thoſe two forces; that is, with a force equal to $F \times AK$, the diſtance of the hand from the point A. So that the ſcale D muſt preponderate, whether the hand be applied to that part of the beam, which lies on the ſame ſide of the axis with the man, or to that which lies on the other ſide; and if D be put to denote the diſtance of that point to which the hand is applied from the point A, the force wherewith the preponderating ſcale deſcends, will be univerſally as $F \times D$; that is, as the force which the hand exerciſes againſt the beam, multiplied into the diſtance of the hand from the point A. And if the force wherewith the hand preſſes the beam be required, it may be diſcovered by throwing in as much weight into the ſcale E, as is ſufficient to balance the force of the hand, and to prevent the deſcent of the ſcale D: for putting W to denote that weight, its moment is as $W \times BC$ or AC, which being equal to $F \times D$, the moment of EF will be found equal to $W \times \frac{AC}{D}$; that is, to the weight multiplied into half the length of the beam, and divided by the diſtance of the hand from A. For inſtance; If the balancing weight be twenty pounds, and the diſtance of the hand from A be to half the length of the beam, as one to two, the force wherewith the hand preſſes the beam, is equal to twenty pounds, multiplied by two, and divided by unity; that is, it is equal to forty pounds. From what has been ſaid, it follows, that

when the hand is applied to that part of the beam, which lies on the ſame ſide of the axis with the man, the force of the hand upon the beam is greater than the weight which balances it in the ſcale E, and leſs than the ſame when the hand is applied to that part of the beam, which lies on the other ſide of the axis with reſpect to the man; for, in the firſt caſe, $W \times \frac{AC}{D}$ is greater than W, and in the latter leſs, inaſmuch as AC is in the former caſe always greater, and in the latter leſs than D.

Note, All that the Doctor has here advanced, eaſily follows from Corol. 1. to Problem XXV.

In the Philoſophical Tranſactions, N°. 407, Doctor *Deſaguliers*'s explication of the above-mentioned property of the balance is as follows:



A B is a balance, on which is ſuppoſed to hang at one end B, the ſcale E, with a man in it, who is counterpoiſed by the weight W, hanging at A, the other end of the balance. I ſay, that if ſuch a man with a cane, or any rigid ſtrait body, puſhes upwards againſt the beam, any where between the points C and B (provided he does not puſh directly againſt B) he will thereby make himſelf heavier, or overpoiſe the weight W, though the ſtop G G hinders the ſcale E from being thruſt outwards from C towards GG. I ſay likewiſe, that if the ſcale and man ſhould hang from D, the man by puſhing upwards againſt B, or any where between B and D (provided he does not puſh directly againſt D) will make himſelf lighter, or be overpoiſed by the weight W, which did before only counterpoiſe the weight of his body and the ſcale.

If the common center of gravity of the ſcale E, and the man ſuppoſed to ſtand in it be at *k*; and the man, by thruſting againſt any part of the beam, cauſe the ſcale to move outwards, ſo as to carry the ſaid common center of gravity to *k x*; then, inſtead of BE, LI will become the line of direction of the compound weight, whoſe action will be increaſed in the ratio of LC to BC. This is what has been explained by ſeveral writers of mechanics, but no one, that I know of, has conſidered the caſe when the ſcale is kept from flying out, as here by the poſt GG, which keeps it in its place, as if the ſtrings of the ſcale were become inflexible. Now to explain this caſe, let us ſuppoſe the length BD of half of the brachium BC to be equal to three feet, the line BE to four feet, the line ED of five feet, to be the direction in which the man puſhes; DF and FE to be reſpectively equal and parallel to BE and BD; and the whole or abſolute force with which the man puſhes, equal to (or able to raiſe) ten ſtone. Let the oblique force ED (=10 ſtone) be reſolved into the two EF and EB (or its equal FD) whoſe directions are at right angles to each other, and whoſe reſpective quantities (or intenſities) are as 6 and 8; becauſe EF and BE are in that proportion to each other, and to ED. Now ſince EF is parallel to BDCA, the beam, it does no way affect the beam to move it upwards; and therefore there is only the force repreſented by FD, or eight ſtone, to puſh the beam upwards at D. For the ſame reaſon, and becauſe action and reaction are equal, the ſcale will be puſhed down at E, with the force of eight ſtone alſo. Now ſince the force at E, pulls the beam perpendicularly downwards from the point B, diſtant from C the whole length of the brachium BC, its action downwards will not be diminiſhed, but may be expreſſed by $8 \times BC$; whereas the action upwards againſt D will be half loſt, by reaſon of the diminiſhed diſtance from the center, and is only to be expreſſed by $8 \times \frac{BC}{2}$; and when the action upwards to raiſe the beam, is ſubtracted from the action downwards to depreſs it, there will ſtill remain four ſtone to puſh down the ſcale; becauſe $8 \times BC - 8 \times \frac{BC}{2} = 4\,BC$; conſequently, a weight of four ſtone muſt be added at the end A, to reſtore the equilibrium. *Therefore a man*, &c. *puſhing upwards under the beam, between* B *and* D, *becomes heavier*. Q. E D. On the contrary; If the ſcale ſhould hang at F, from the point D, only three feet from the center of motion C, and a poſt *g g* hinders the ſcale from being puſhed inwards towards C; then if a man in this ſcale F, puſhes obliquely againſt B with the oblique force above-mentioned, the whole force, for the reaſons before given (in reſolving the oblique force into two others, acting in lines perpendicular to each other) will be reduced to eight ſtone, which puſhes the beam directly down at D towards F. But as CD is only equal to half of CB, the force at D, compared with that at

B,

B, loſes half its action, and therefore can only take off the force of four ſtone from the puſh upwards at B; and conſequently the weight W at A will preponderate, unleſs an additional weight of four ſtone be hanged at B. *Therefore a man*, &c. *puſhing upwards under the beam, between* B *and* D, *becomes lighter.*

Schol. 1. Hence, knowing the abſolute force of the man that puſhes upwards (that is, the whole oblique force) the place of the point of truſion D, and the angle made by the direction of the force, with a perpendicular to the beam at the ſame point; we may have a general rule to know what force is added to the end of the beam B, in any inclination of the direction of the force or place of the point D.

RULE *for the firſt caſe.* Firſt find the perpendicular force by the following analogy, whoſe demonſtration is known to all that underſtand the application of oblique forces.

As the radius :
To the right ſine of the angle of inclination : :
So is the oblique force :
To the perpendicular force.

Then the perpendicular force being multiplied in the length of the brachium BC, *minus* the ſaid force, multiplied into the diſtance DC, will give the value of the additional force at B, or of the weight required to reſtore the equilibrium at A.

Or to expreſs it in the algebraical way. Let of expreſs the oblique force, pf the perpendicular force, and x the force required, or value of the additional weight at A, to reſtore the equilibrium.

$$\overline{DE : DF} : : \overline{of : pf}$$
$$\overline{pf \times BC} - \overline{pf \times DC} = x.$$

The ſame rule will ſerve for the ſecond caſe, if the quantity found be made negative, and the additional weight ſuſpended at B. Or, having found the value of the perpendicular force, the equation will ſtand thus— $\overline{pf \times BC} + \overline{pf \times DC} = -x$; and conſequently the additional weight muſt be hanged at B, becauſe $-x$ at A is the ſame as $+x$ at B.

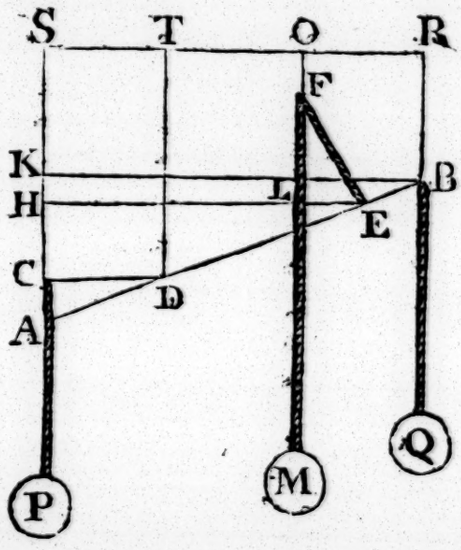
Schol. 2. Hence it follows alſo, that if, in the firſt caſe, the point of truſion be taken at C, the force at B, (or force whoſe value is required) will be the whole perpendicular force; becauſe CD is equal to nothing. And if the point D be taken beyond C towards A, the perpendicular force

puſhing upwards at that point, multiplied into DC, muſt be added to the ſame force, multiplied into BC; that is, $\overline{pf \times BC} + \overline{pf \times DC} = x$.

PROBLEM XXXIV.

LET ADB be an inflexible rod or lever, whoſe center of motion is D; to the extremities A and B of this lever, ſuppoſe the weights P and Q freely ſuſpended; and from a given point E in AB, let a weight M act in the direction EF, by means of a ſtring faſtened at E, and paſſing over a ſmall pulley fixed in the point F; it is required to find the poſition of the lever, when the weights are ſuſtained in equilibrio.

SOLUTION.



Imagine AB the required ſituation of the rod. Through any point above F, as O, draw SOR parallel to the horizon; draw alſo to SR, from the points A, D, F and B, the perpendiculars AS, DT, FO and BR. From D, E and B, draw DC, EH and BK, parallel to SOR. Put $AD = a$, $DE = b$, $EB = c$, $DT = g$, $OF = e$, $TO = f$, $a + b + c = s$, $a + b = p$, L, l and m for the reſpective lengths of the ſtrings AP, EFM and BQ, $ST = CD = x$; then, $AS = \sqrt{a^2 - x^2} + g$, $AK = \frac{s\sqrt{1 - xx}}{a}$, $BR = g + \sqrt{a^2 - x^2} - \frac{s\sqrt{1 - xx}}{a}$, as $a : x :: p : \frac{px}{a} = HE$, whence $AH = \sqrt{p^2 - \frac{p^2 x^2}{a^2}}$; conſequently $LE = \frac{px}{a} - x - f$, and $FL = \sqrt{a^2 - x^2} + g - \sqrt{p^2 - \frac{p^2 x^2}{a^2}} - e$, therefore $FE = \sqrt{\sqrt{a^2 - x^2} - \sqrt{p^2 - \frac{p^2 x^2}{a^2}} + g - e}^2 + \overline{\frac{px}{a} - x - f}^2$. Hence the perpendicular diſtance of the common center of gravity of the three weights P, M and Q, from the horizontal line SR, will be expreſſed by

$$\frac{P\times L+P\times\sqrt{a^2-x^2}+P\times g+Q\times\sqrt{a^2-x^2}+Q\times g-Q\times\frac{s\sqrt{a^2-x^2}}{a}+Q\times m}{P+Q+M}$$

$$+\frac{e\times M+M\times l-M\times\sqrt{\sqrt{a^2-x^2}-\sqrt{p^2-\frac{p^2x^2}{a^2}}+g-e+\frac{px}{a}-x-f}^2}{P+Q+M};$$

which being put into fluxions, made equal to nothing, and the equation thence arising properly reduced, will give $P\times\frac{CD}{AC}-Q\times\frac{CD}{AC}\times\frac{DB}{AD}$

$$=M\times\frac{FL\times CD}{FE\times AC}-\frac{FL\times AE^2\times CD}{FE\times AH\times AD^2}\times M-\frac{LE\times DE}{FE\times AD}\times M.$$

COROL. 1. When ADB obtains a direction parallel to the horizon, AD will coincide with CD, AC, AH and AK will entirely vanish, and the general equation become $M\times\frac{LE\times DE}{FE}=Q\times DB-P\times AD.$

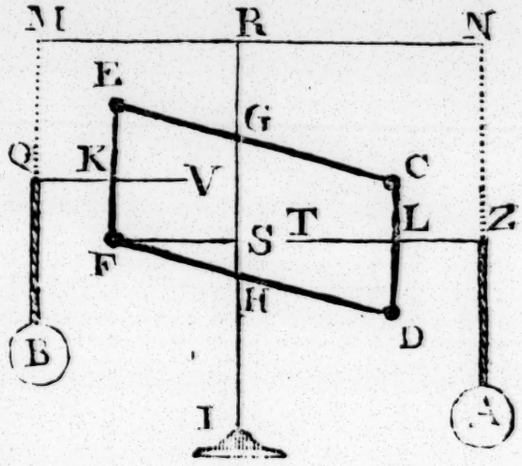
COROL. 2. If $M=0$, we shall have $Q\times DB=P\times AD$, whence $Q:P::AD:DB$. By the first corollary it appears, that $P\times AD+M\times DE\times\frac{LE}{FE}=Q\times DB$; and hence we have an easy method (deduced from our principles) of determining what Doctor *Desaguliers* has proposed in the first scholium to the preceding discourse, *viz.* to find what force is to be added to the end of the beam B, (see *page* 79) in any inclination of the direction of the force or place of the point D, to restore the equilibrium. For FE : EL (in the last figure) or radius : sin $\angle$ EFL :: $M\times DE$: the quantity required, which must be added to the momentum of P, to restore the former momentum of Q.

PROBLEM XXXV.

LET the solid and inflexible sides of the parallelogram FECDF be supposed moveable about its angular points F, E, C, D, and the opposite sides EC, FD, also moveable about two pins, fixed at G and H in the vertical line IG. At right angles to the sides FE and DC of this parallelogram, two cross pieces, as QV, TZ, (which are here considered as inflexible rods of equal lengths and weights) are fastened. Let now two weights B and A be freely suspended from the points Q and Z; it is re-

quired to find the poſition of the ſaid parallelogram, whoſe ſides EF and DC, do, by the motion thereof, aſcend or deſcend in directions perpendicular to the horizon, when thoſe weights A and B are a juſt counterpoiſe to each other.

SOLUTION.



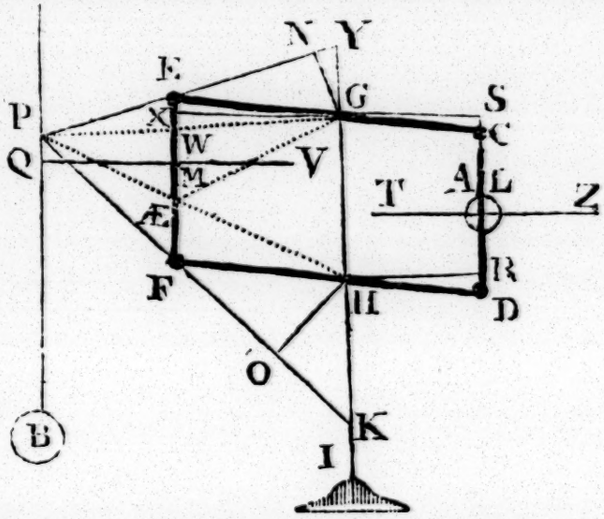
Imagine the poſition in the figure to be the required one. Draw a right line parallel to the horizon above EC, as MN; produce BQ, HG and AZ, to M, R and N. From F, draw FS perpendicular to HG; and put $HF = GE = a$, $HD = GC = b$, $FK = c$, $DL = d$, $HR = p$, the ſtrings QB, ZA, equal to r and s reſpectively, and x for the ſine of the angle HFS, radius $= 1$; then will $HS = ax$, and conſequently $BM = p - ax - c + r$, and $AN = p + bx - d + s$; therefore, by Lemma 1, $\frac{pB - Bax - Bc + Br + pA + bAx - Ad + As}{A + B}$ expreſſes the perpendicular diſtance of the common center of gravity of the weights A, B, from the horizontal line MRN, which muſt be the greateſt poſſible; and therefore its fluxion, *viz.* $-Ba\dot{x} + bA\dot{x} = 0$, hence $Ba = bA$, or $B : A :: b : a$.

From this inveſtigation it appears, that, provided the weights are reciprocally as the diſtances of the points F and D from the axis of motion, an equilibrium will enſue, be thoſe weights ſuſpended from any points at pleaſure in the croſs pieces QV, TZ. Nor will the equilibrium ſo obtained be at all affected by the poſition of the parallelogram ECDFE; for in the equation which reſults from the above expreſſion, being put into fluxions, and made equal to nothing, the quantities p, s, c, r, d and x, have no place; and conſequently the weights will, if once in equilibrio, conſtantly remain ſo, let thoſe quantities p, s, c, &c. be what they will.

The machine deſcribed in this problem, is ſaid to have been the invention of M. *Roberval:* and the ſurpriſing property of it, which we have juſt now inveſtigated by an eaſy proceſs deduced from our general principle, was many years ſince demonſtrated by M. *Parent*, in the third volume of his *Eſſais & Reſcherches de Mathematique*, and of which the following is an exact tranſlation:

" This

" This machine is a parallelogram CEFD, whoſe ſides are ſolid and moveable about the angular points. It is faſtened by two points G and H, taken upon the ſides CE, DF, at equal diſtances from CD, EF, and in the vertical rod GHI; about which, by means of two pins faſtened in G and H, it freely turns. The tranſverſe pieces ZT, VQ are fixed to the ſides CD, EF, in the points L and M. Theſe tranſverſe pieces may be of any length at pleaſure, but their weights muſt be in the reciprocal ratio of the diſtances GX, GS, of thoſe ſides CD, EF. The property of this admirable balance is ſuch, that if two weights A, B, are freely ſuſpended from the tranſverſe pieces ZT, VQ, in any points thereof, they will remain in equilibrio, provided thoſe weights are reciprocally as the ſaid diſtances GS, GX.



It is evident, that if the parallelogram CEFD be turned about the fixed points H, G, the weight A will aſcend or deſcend, juſt in the ſame manner as it would if faſtened to either of the ſides CD, EF of the parallelogram; for, by the conſtruction of this machine, thoſe ſides muſt aſcend or deſcend vertically, in the ſame manner with the tranſverſe beams ZT, VQ; and therefore if we imagine the weights A, B, faſtened upon CD, EF, reſpectively as in L, M, their aſcents or deſcents made in the ſame time, will be conſtantly in the ratio of the diſtances GS, GX, of thoſe ſides from the axis GH, or reciprocally as the weights. This being premiſed, let us ſuppoſe the weight A (for example) faſtened upon CD at the point L, to be a juſt counterpoiſe to the weight D, ſuſpended from any point Q in PB. In a ſtate of reſt, the weight B endeavours to turn the ſide EF, from E towards H, and is hindered therefrom by the pins EF, about which the ſides of the parallelogram move; the pin E keeps the point E from moving in the direction FP, and the other pin F keeps the point F from moving in the direction FP, which cuts NEP in P. With regard to the weight A, we may conſider one part thereof as ſuſpended in C, and the remaining part in D. We have now the two levers CGE, DHF, whoſe centers of motion are G and H, each being charged with a part of the weight A in CD; the firſt by means of a power or force derived from B, which draws the point E according to the direction NEP; and the ſecond by means of ano-

ther force derived alſo from B, which puſhes the point F, according to the direction PFO, in ſuch manner, that, drawing the perpendiculars GN, HO, to NEP and OFP reſpectively, and the perpendicular HR upon CD, the force acting in the direction EP, being multiplied by GN, muſt be equal to the firſt part of the divided force or weight A, multiplied by GS; and, at the ſame time, the force acting in the direction PF, multiplied by HO, muſt be equal to the other part of A, multiplied by HR. Moreover, as the weight B makes an equilibrium with the two reſiſtances in E and F, the firſt of which acts in the direction PE, and the ſecond reacts in the direction EP, it is manifeſt, that the three directions PE, PF, QB, will interſect each other in one and the ſame point P upon QB; and that EF, being parallel to QB, the three ſides of the triangle EFP will denote the three powers; that is, EF will repreſent the weight B, EP the reſiſtance of E, and FP that of F. But as the directions PE, PF, are unknown, it is evident we may take P upon QB at pleaſure; from whence to draw PE, PF, and that always the product of PE by GN, ſhall be equal to that of one part of A by GS (EF always repreſenting B) and at the ſame time the product of PF by HO, ſhall likewiſe be equal to the remaining part of A, multiplied by HR.

It remains now to prove, that when theſe conditions take place, the weight B will be to the weight A reciprocally, as SG : GK, the ſame as if the weight B was faſtened upon EF; or that B, multiplied by GX, ſhall give the ſame momentum as A, multiplied by SG; or taking EF in the room of B, and the two products of PE by GN, and PE by HO, inſtead of the ſingle product of A by SG (which was proved to be equal to the other two products taken together) the product of EF by GX is always equal to the ſum of the products of EF by GN, and PF × HO; or drawing the right lines GP, HP, cutting EF in W and Æ, we muſt prove, that the triangle GEF is always equal to the ſum of the two triangles GEP, HPE. In order to do this, draw the right lines GF, GÆ, and produce PN, PO, to meet HG and GH in Y and K. The triangles GFÆ, HFÆ, are equal, being upon the ſame baſe, and between the ſame parallels; from the triangle GEF, take the triangles GEW, GFÆ, or HFÆ, and it remains only to prove, that the triangle GWÆ is always equal to the ſum of the triangles EDW and FPÆ; or, which comes to the ſame thing, that FW + EF : WÆ :: GX : MQ, which may be ſhewn thus: Becauſe EF is parallel to YK, we have YK : EF :: YP : EP, therefore YK — EF : EF :: EY : EP :: GX : MQ, and conſequently EW + ÆF : WÆ : GX : MQ; it is therefore evident, that B is to L, when in equilibrio, as GS to GX reciprocally, in the ſame manner as if the weight B were ſuſpended at E; it alſo follows

from hence, that the weight B, being ſuſpended from any point in VQ, taken at pleaſure, will be a juſt counterpoiſe to the other weight A."

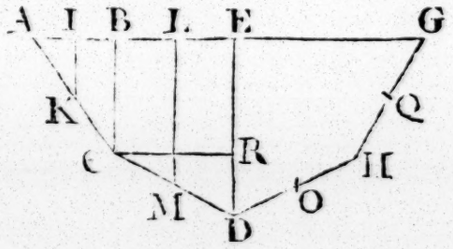
Notwithſtanding M. *Parent* has in the inveſtigation before us clearly deduced the propoſed property of *Roberval*'s balance, by the application of the method of dividing forces; yet the whole proceſs appears much inferior to the ſolution we have given by that general property of the common center of gravity of bodies, ſo conſtantly applied to the ſolutions of the various problems contained in this treatiſe.

PROBLEM XXXVI.

FROM the given points A, G, in the horizontal line AG, let four beams AC, CD, DH and HG, joined together at C, D, H, and moveable about the points A, C, D, H, G, be ſuſpended. It is required to determine the poſition in which thoſe beams will be ſuſtained at reſt, ſuppoſing they are of an equal and given length, their centers of gravity in the middle points K, M, O, Q, and the given weights of AC, CD, reſpectively equal to thoſe of GH and HD.

SOLUTION.

Suppoſe ACDHG the required poſition. From the points K, C, M, D, *&c.* draw KI, CB, ML, DE; and perpendicular to the horizontal line AG, draw CR parallel to AG. Put AC $=$ CD $= 2d$, AE $= n$, AI $= x$, and W for the weight of AC or HQ, w for the weight of CD or DH; then IK $= \sqrt{d^2 - x^2}$, BC $= 2\sqrt{d^2 - x^2}$, $n - 2x$ $=$ CR, and $\sqrt{4d^2 - n - x^2}$ $=$ DR,



whence $2\sqrt{d^2 - x^2} + \sqrt{4d^2 - n - 2x^2} =$ ED, $\frac{4\sqrt{d^2 - x^2} + \sqrt{4d^2 - n - 2x^2}}{2}$ $=$ ML; conſequently $\frac{2W \times \sqrt{d^2 - x^2} + 4w\sqrt{d^2 - x^2} + w\sqrt{4d^2 - n - 2x^2}}{2W + 2w}$ expreſſes the diſtance of the common center of gravity of all the beams from the horizontal line AG, which muſt be a maximum; therefore $2W \times \frac{-x\dot{x}}{\sqrt{d^2 - x^2}} + 4w \times \frac{-x\dot{x}}{\sqrt{d^2 - x^2}} + w \times \frac{\overline{n - x} \times 2\dot{x}}{\sqrt{4d^2 - n - 2x}} = 0$; hence

we get $W \times \frac{AI}{KI} + 2w \times \frac{AI}{KI} = w \times \frac{CR}{DR}$, or $\overline{W + 2w} \times \frac{AB}{CB} = w \times \frac{CR}{DR}$.

If the points A, C, D, H, G, are in the periphery of a ſemicircle, then AE = ED; and becauſe AC = CD, by the problem, the angle AEC will be 45 degrees; conſequently the angle ACB 22°. 30′, and CDR 67°. 30′; $\frac{AB}{CB}$ and $\frac{CR}{DR}$ are the tangents of the angles ACB, CDR, reſpectively, radius being unity; hence $\overline{W + 2w}$ × tangent of 22°. 30′. = w × tangent of 67°. 30′; that is, $\overline{W + 2w} \times 4142135 = w \times 24142135$, and therefore $W = w \times 3{,}82$.

Let the points A, C, D, H, G, be ſituated in the periphery of a ſemi-ellipſis, whoſe tranſverſe diameter is AG; ſemiconjugate diameter ED, to determine the ratio of W to w. Put AE = t, ED = c, EB = CR = x; then will $BC^2 = \frac{cc}{tt} \times \overline{tt - xx}$, and $AC^2 = \frac{cc}{tt} \times \overline{tt - xx} + \overline{t - x^2}$; but $x^2 + \overline{c - \frac{c}{t} \times \sqrt{t^2 - x^2}}^2 = CD^2 = AC^2$, therefore $x^2 + c^2 - \frac{2c^2}{t}\sqrt{t^2 - c^2} = \overline{t - x^2}$; hence $x = \sqrt{\frac{m^2 n^2}{n^2+1^2} + \frac{tt - m^2}{n^2+1}} - \frac{mn}{n^2+1}$, wherein $m = \frac{tc^2 - t^3}{2c^2}$, and $n = \frac{t^2}{c^2}$. Let this value of x be ſubſtituted in the equation $\overline{W + 2w} \times \frac{t - x}{\frac{c}{t} \times \sqrt{t^2 - x^2}} = w \times \frac{x}{c - \frac{c}{t} \times \sqrt{t^2 - x^2}}$, which is the ſame with $\overline{W + 2w} \times \frac{AB}{CB} = w \times \frac{CR}{DR}$, and thence the ratio of W to w may be determined in terms of t and c. Suppoſe, for example, $t = 2c$, then will $x = 1{,}157 \times c$, and therefore $\overline{W + 2w} \times 1{,}308 = w \times 3{,}254$; hence W = ,48 w.

Let us now ſuppoſe the points A, C, D, H, G, ſituated in the arch of the common or conic parabola, whoſe axis is ED, correſponding ordinate AG, and parameter p. Put AE = EG = a, ED = b, EB = x, then $AB \times BG = a^2 - x^2$; and, by the property of the curve, $\frac{a^2 - x^2}{p} = BC$, $RD = b - \frac{a^2 - x^2}{p}$; and becauſe AC = CD by the problem, we ſhall have

have this equation, $\overline{a-x^2} + \frac{a^2-x^2}{p^2} = x^2 + \frac{bp-a^2+x^2}{p^2}$; which, properly reduced, gives $x = \frac{a\sqrt{2a^2b^2+2b^4+a^4}-a^3}{2b^2}$. If in the general equation, $\overline{W+2w} \times \frac{AB}{CB} = w \times \frac{CR}{DR}$, we ſubſtitute for AB, CB, CR and DR, their reſpective values, in terms of a, b, p and x, there will reſult $Wx = aw - xw$; and again, for x in this equation, ſubſtitute the above found value thereof, and we ſhall have $\frac{W \times \sqrt{2a^2b^2+2b^4+a^4}-a^2W}{2b^2} = w - \frac{w \times \sqrt{2a^2b^2+2b^4+a^4}+a^2w}{2b^2}$, or $W \times \sqrt{2a^2b^2+2b^4+a^4} - a^2W = 2b^2w + a^2w - w \times \sqrt{2a^2b^2+2b^4+a^4}$, from whence the ratio of W to w may be aſcertained in terms of a and b.

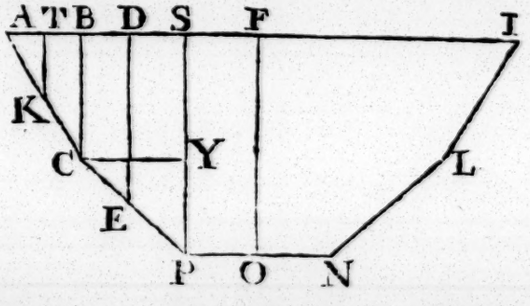
Let $a = b$, then the laſt equation will become $W\overline{\times\sqrt{5}-1} = w \times 3-\sqrt{5}$, whence $W = ,618\,w$.

PROBLEM XXXVII.

OTHER things remaining as in the laſt problem, let there be five equal and ſimilar beams ſuſpended as before; to determine the poſition in which they will be ſuſtained at reſt.

SOLUTION.

Let ACPNLI be the required poſition. Draw from the middle points K, E, O, of the beams AC, CP, PN, the lines KT, ED and OF, each perpendicular to the horizontal line AI; draw alſo CB and PS parallel to DE, and CY to AI; let W repreſent the weight of the beam AC, or its equal IL, w that of CP or NL, and u that of PN. Put AC $= CP = PN = NL = LI = 2d$, $AF = FI = m$, $AS = m - d = n$, $AT = x$; then $AB = 2x$, $BC = 2\sqrt{d^2-x^2}$, $PY = \sqrt{4d^2-\overline{n-2x}^2}$, $DE = \frac{\sqrt{4d^2-\overline{n-2x}^2}+4\sqrt{d^2-x^2}}{2}$,



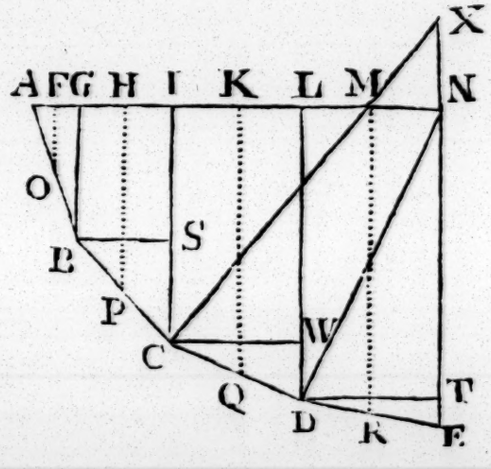
and PS $= \sqrt{4d^2 - \overline{n - x^2}} + 2\sqrt{d^2 - x^2}$. Now by Lemma 1, the perpendicular diſtance of the common center of gravity of all the beams, from the horizontal line AI, will be expreſſed by $\frac{2TK \times W + 2DE \times w + SP(FO) \times u}{2W + 2w + 2u}$. For TK, DE, and SP, in this expreſſion, ſubſtitute their values as above found, we have $\overline{2W + 4w + 2u} \times \sqrt{d^2 - x^2} + \overline{w + u} \times \sqrt{4d^2 - \overline{n - 2x^2}}$; a maximum, which being put into fluxions, and properly reduced, gives $\overline{W + 2w + u} \times \frac{AB}{BC} = \overline{w + u} \times \frac{CY}{PY}$.

If the points A, C, P, N, L, I, are in the periphery of a ſemicircle, whoſe diameter is AI, then the angle CAT will be 72 degrees, and the angle CYP 54 degrees; therefore $\frac{AB}{BC}$ and $\frac{CY}{PY}$ are the tangents of 18 and 54 degrees reſpectively, radius being unity; conſequently $\overline{W + 2w + u} \times 3249166 = \overline{w + u} \times 13763819$; and, by aſſuming the value of one of the quantities, the ratio of the other two may be determined.

PROBLEM XXXVIII.

SUPPOSE eight equal and ſimilar beams are ſuſpended after the manner deſcribed in the two preceding problems, and ABCDENA to repreſent one half of the figure they will form when in the required poſition, and which we are now to determine.

SOLUTION.



Let the middle points O, P, Q, R, be the centers of gravity of the beams AB, BC, CD, DE; and alſo let thoſe letters repreſent their weights reſpectively. Draw OF, BG, PH, *&c.* perpendicular to the horizontal line AN; and BS, CW, DT, parallel to it. Now it is evident, by Lemma 1, that $\frac{2FO \times O + 2HP \times P + 2KQ \times Q + 2MR \times R}{2O + 2P + 2Q + 2R}$ is the perpendicular diſtance of the common center of gravity of the eight beams, from the horizontal line AN; therefore the numerator of this

fraction, or, which comes to the ſame thing, $\overline{O+P} \times BG + \overline{P+Q} \times CI + \overline{Q+R} \times LD + NE \times R$, muſt be the greateſt poſſible. This being premiſed, put $AN = n$, $AB = BC = CD = DE = d$, $AG = x$, $GI = y$, $IL = z$; then $BG = \sqrt{d^2 - x^2}$, $CI = \sqrt{d^2 - x^2} + \sqrt{b^2 - y^2}$, $DL = \sqrt{d^2 - x^2} + \sqrt{b^2 - y^2} + \sqrt{d^2 - z^2}$, and $EN = \sqrt{d^2 - x^2} + \sqrt{b^2 - y^2} + \sqrt{d^2 - z^2} + \sqrt{d^2 - n - x - y - z^2}$; therefore $\overline{O+P} \times \sqrt{d^2 - x^2} + P + Q \times \overline{\sqrt{d^2 - x^2} + \sqrt{b^2 - y^2}} + \overline{Q+R} \times \overline{\sqrt{d^2 - x^2} + \sqrt{b^2 - y^2} + \sqrt{d^2 - z^2}} + R \times \overline{\sqrt{d^2 - x^2} + \sqrt{b^2 - y^2} + \sqrt{d^2 - z^2} + \sqrt{d^2 - n - x - y - z^2}}$, a maximum. In fluxions with x only flowing the reſt conſtant gives, when properly reduced, $\overline{O+P} \times \frac{AG}{BG} + \overline{P+Q} \times \frac{AG}{BG} + \overline{Q+R} \times \frac{AG}{BG} + R \times \frac{AG}{BG} = R \times \frac{DT}{ET}$, or $\overline{O+2P+2Q+2R} \times \frac{AG}{BG} = R \times \frac{DT}{ET}$; and by taking the fluxion of the expreſſion for the maximum, with x and z conſtant, we ſhall have $\overline{P+Q} \times \frac{GI}{SC} + \overline{Q+R} \times \frac{GI}{SC} + R \times \frac{GI}{SC} = \frac{LN}{ET} \times R$, hence $\frac{DT}{ET} \times R = \overline{P+2Q+2R} \times \frac{BS}{SC}$; and laſtly, by making z flow, the fluxion of the ſaid expreſſion will, after due reduction being made, give $\frac{DT}{ET} \times R = \overline{Q+R} \times \frac{IL}{DW} + R \times \frac{IL}{DW}$, whence $\overline{Q+2R} \times \frac{CW}{DW} = \frac{DT}{ET} \times R$. From theſe three equations, *viz.*

$$\left.\begin{aligned} \overline{O+2P+2Q+2R} \times \frac{AG}{BG} \\ \overline{P+2Q+2R} \times \frac{BS}{CS} \\ \overline{Q+2R} \times \frac{CW}{DW} \end{aligned}\right\} = \frac{DT}{ET} \times R.$$

by aſſuming the value of R, the values of Q, P and O, may be eaſily determined; and hence the law of continuation, by which the ratio of the weights may be found, becomes manifeſt, let the number of beams ſuſpended between the points A and E, be what they will: for ſuppoſe O, P, Q, R, S, T, U, *&c.* repreſent the reſpective weights, placed at the

middle points O, P, Q, R, &c. of the beams, the general equations will then be

$$\left.\begin{array}{l} \frac{AG}{BG} \times \overline{O+2P+2Q+2R+2S, \&c.} \\ \frac{BS}{CS} \times \overline{\ast \quad P+2Q+2R+2S, \&c.} \\ \frac{CW}{DW} \times \overline{\ast \quad \ast \quad Q+2R+2S, \&c.} \\ \quad \&c. \qquad \&c. \qquad \&c. \end{array}\right\} = \frac{DT}{ET} \times S, \&c.$$

If the points A, B, C, D and E, are placed in the arch of a circle; that is, if AN (= NE) is the radius of a circle, whoſe periphery paſſes through the points B, C, D; then, by the property of that curve, the angle ABG will be 11°. 15′, BCS will be 33°. 45′, CDW 56°. 15′, and DET 78°. 45′, whence the ratio of O to P, P to Q, &c. may be eaſily determined.

PROBLEM XXXIX.

ALL other things remaining as in the laſt problem, let the weights O, P, Q and R, be placed at the angular points B, C, D and E, inſtead of the centers of gravity O, P, Q, &c. of the beams, which are here conſidered only as ſtrings that ſerve to keep the weights in equilibrio; it is required to find the poſition of thoſe weights, when ſuſtained at reſt or in equilibrio among themſelves.

SOLUTION.

It is evident by Lemma 1, that $\frac{2BG \times O + 2IC \times P + 2LD \times Q + NE \times R}{2O+2P+2Q+R}$ is the perpendicular diſtance of the common center of gravity of all the weights from the horizontal line AN. Subſtitute, as before, for BG, IC, &c. and we ſhall have $2O \times \sqrt{d^2-x^2} + 2P \times \sqrt{d^2-x^2} + 2P \times \sqrt{d^2-y^2} + 2Q \times \sqrt{d^2-x^2} + 2Q \times \sqrt{b^2-y^2} + 2Q \times \sqrt{d^2-z^2} + R \times \sqrt{d^2-x^2} + R \times \sqrt{b^2-y^2} + R \times \sqrt{d^2-z^2} + R \times \sqrt{d^2-n-x-y-z^2}$, which muſt be a maximum. In fluxions, with y and z conſtant, gives $\overline{2O+2P+2Q+R} \times \frac{AG}{BG} = \frac{DT}{ET} \times R$; and by taking the fluxion of the ſame expreſſion, with x

and

and z conſtant, we ſhall have $\overline{2Q+2P+R} \times \frac{BS}{CS} = \frac{DT}{ET} \times R$; and laſtly, by making z flow, we ſhall have $\overline{2Q+R} \times \frac{CW}{DW} = \frac{DT}{ET} \times R$; hence $2Q : R :: \tan \angle DET - \tan \angle CDW : \tan \angle CDW$, and $2P : 2Q+R :: \tan \angle CDW - \tan \angle BCS : \tan \angle BCS$; alſo $2O : 2P+2Q+R :: \tan \angle BCS - \tan \angle ABG : \tan \angle ABG$.

If the points A, B, C, D and E, are ſituated in an arch of a circle, whoſe radius is AN (= NE) then the general equations will become

$$\left.\begin{array}{rl} \overline{2O+2P+2Q+R} \times & ,1989123 \\ \overline{2P+2Q+R} \times & ,6681786 \\ \overline{2Q+R} \times & 1,4966057 \end{array}\right\} = R \times 5,0273394.$$

whence we get $Q = R \times 1,142447$. $P = R \times 2163003$.

By the preceding general equations, *viz.*

$$\left.\begin{array}{r} \overline{2O+2P+2Q+R} \times \frac{AG}{BG} \\ \overline{2P+2Q+R} \times \frac{BS}{SC} \\ \overline{2Q+R} \times \frac{CW}{DW} \end{array}\right\} = \frac{DT}{ET} \times R,$$

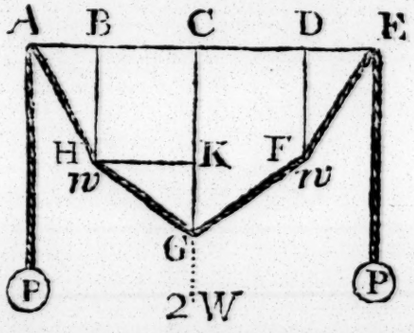
the ratio of O, P, Q, R, may be determined, for $2Q+R = \frac{DT}{ET} \times \frac{DW}{CW} \times R$; therefore $\overline{2P + \frac{DT}{ET} \times \frac{DW}{CW} \times R} \times \frac{BS}{SC} = \frac{DT}{ET} \times R$, hence $2P = R \times \frac{SC}{BS} \times \frac{DT}{ET} - \frac{DT}{ET} \times \frac{DW}{CW}$, and $\overline{\overline{2O+R \times \frac{SC}{BS} \times \frac{DT}{ET} - \frac{DT}{ET} \times \frac{DW}{CW}} + R \times \frac{DT}{ET} \times \frac{DW}{CW}} \times \frac{AG}{BG} = \frac{DT}{ET} \times R$; from hence we get the following ratios, $2Q : R :: \frac{DT}{ET} - \frac{CW}{DW} : \frac{CW}{DW}$, or, which is the ſame thing, $2Q : R :: \frac{DW}{CW} - \frac{ET}{DT} : \frac{ET}{DT}$; $2P : R :: \frac{SC}{SB} - \frac{DW}{CW} : \frac{ET}{DT}$, and $2O : R :: \frac{BG}{AG} - \frac{SC}{SB} : \frac{ET}{DT}$.

Therefore if at the points D, C, B, *&c.* perpendiculars be erected to DC, CB, *&c.* reſpectively; the weights to be placed at thoſe points, in order to preſerve the equilibrium, will be directly as the difference of the tangents of the angles which thoſe perpendiculars make with the line EN. For ſuppoſe DN and CX drawn perpendicular to ED and DC reſpectively, then becauſe $2Q : R :: \frac{DW}{CW} - \frac{ET}{DT} : \frac{ET}{DT}$, $2Q : R ::$ tan $\measuredangle$ DCW — tan $\measuredangle$ EDT : tan $\measuredangle$ EDT, radius being unity; but $\measuredangle$ DCW = $\measuredangle$ CXE, and $\measuredangle$ EDT = $\measuredangle$ DNE; and ſince this holds equally true of the ratio of the other weights, as of 2P to R, or of 2O to R, *&c.* it follows, that the weights are as the difference of the tangents of the angles, which the correſponding perpendiculars make with the axis of the figure.

In this and the three preceding problems, it will make no manner of alteration in their ſolutions, if the beams, inſtead of being ſuſpended, were to ſupport themſelves in a vertical poſition, by means of the weights, placed at their centers of gravity, above the horizontal line; or, which is the ſame thing, if the figures were inverted and placed in a plane perpendicular to the horizon, becauſe the expreſſion for the maximum in one caſe, and the minimum in the other, will be exactly the ſame.

PROBLEM XL.

THE ſtring PAHGFEP ſlides freely over two ſmall pullies, placed at the points A, E, in the horizontal line AE; at the middle point G, between the equal weights P, P, (which are appended to the ends of the ſtrings) a given weight 2W is faſtened; and at H and F, equidiſtant from G, are alſo faſtened the equal weights *w*, *w*. It is required to find the poſition of the ſtring, when the weights P, P, ſuſtain in equilibrio the other weights *w*, 2W, *w*.



SOLUTION.

Suppoſe it done, and AHGFE the ſituation required. Draw BH, GC, FD, perpendiculars to AE, and HK parallel to AE. Put HG = GF = *m*, AC = *n*, *l* for the length of each of the equal ſtrings HAP, FEP, AB = *x*, AH = *z*; then, by Lemma 1, the perpendicular diſtance of

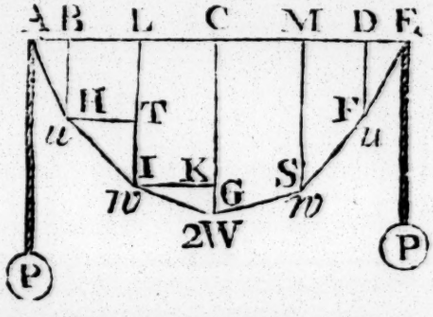
the common center of gravity of all the weights from the horizontal line AE, will be $\frac{2lP - 2Pz + 2w\sqrt{z^2-x^2} + \sqrt{m^2-n-x^2} + \sqrt{z^2-x^2} \times 2W}{2P+2w+2W}$; this expreſſion being put into fluxions, with x conſtant, gives $-2P\dot{z} + 2w \times \frac{z\dot{z}}{\sqrt{z^2-x^2}} + 2W \times \frac{z\dot{z}}{\sqrt{z^2-x^2}} = 0$, whence $P \times BH = \overline{w+W} \times AH$. Let the ſame expreſſion be again put into fluxions, with z conſtant, and we ſhall have $-\overline{w-W} \times \frac{x\dot{x}}{\sqrt{z^2-x^2}} + W \times \frac{n\dot{x} - x\dot{x}}{\sqrt{m^2-n-x^2}} = 0$, whence we get $\overline{w+W} \times \frac{AB}{BH} = W \times \frac{BC}{GK}$; therefore $w + W : W :: \frac{BC}{GK} : \frac{AB}{BH}$ $\therefore w + W : W :: \tan \sphericalangle HGK : \tan \sphericalangle AHB$. But $\tan \sphericalangle HGK : \tan \sphericalangle AHB :: \frac{1}{\tan \sphericalangle GHK} : \frac{1}{\tan \sphericalangle HAB}$; whence it follows, that $w + W : W :: \tan \sphericalangle HAB : \tan \sphericalangle GHK$, and conſequently $w : W :: \tan \sphericalangle HAB - \tan \sphericalangle GHK : \tan \sphericalangle GHK$.

PROBLEM XLI.

OTHER things remaining as in the laſt problem, let five weights be ſuſtained in equilibrio, by means of the equal weights P, P, to determine their poſition when counterpoiſed by the weights u, w, 2 W, *&c.*

SOLUTION.

Draw as before HB, IL, GC, *&c.* perpendiculars to the horizontal line AE, and HT, IK, parallel to it. Put $HI = IG = GS = SF = m$, $AC = n$, l for the length of each ſtring HAP, FEP, $AB = x$, $AH = z$, $BL = y$; then $LC = n - x - y$, and $BH = \sqrt{z^2-x^2}$, $LI = \sqrt{z^2-x^2} + \sqrt{m^2-y^2}$, $CG = \sqrt{z^2-x^2} + \sqrt{m^2-y^2} + \sqrt{m^2-n-x-y^2}$; therefore the perpendicular diſtance of the common center of gravity of all the weights from the horizontal line AE, will, by

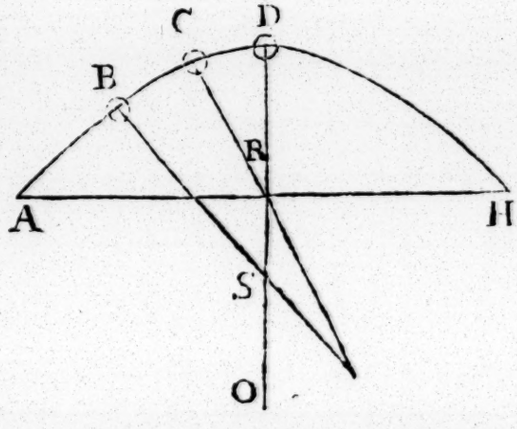


Lemma 1, be expreſſed by $\frac{2u\times\sqrt{z^2-x^2}+2w\times\sqrt{z^2-x^2}+\sqrt{m^2-y^2}}{2P+2u+2w+2W}+$ $\frac{2W\times\sqrt{z^2-x^2}+\sqrt{m^2-y^2}+\sqrt{m^2-n-x-y^2}+2P\times\overline{l-z}}{2P+2W+2u+2w}$; which being put into fluxions with x and z conſtant gives, after due reduction, $\overline{w+W}\times\frac{HT}{IT}=W\times\frac{KI}{GK}$; the ſame expreſſion being put into fluxions, with x and y conſtant, produces this equation, $P\times BH=\overline{u+w+W}\times AH$; and laſtly, by making x flow, we ſhall have $\overline{u+w+W}\times\frac{AB}{BH}=\frac{IK}{GK}\times W$. Hence the law of continuation becomes known, let the number of weights ſuſpended (under the conditions mentioned in the problem) between the points A and G, be what they will.

Corol. 1. The ſum of all the weights ſuſpended between the points A and E, over which the threads or ſtrings freely ſlide, is to the two ſuſtaining weights 2 P, as the ſine of the angle which the ſtring makes with the horizontal line AE to radius; and conſequently the ſum of thoſe weights is as the ſecant of the angle, which that line makes with A P, a perpendicular to the horizon.

Corol. 2. $2P:2W::\frac{IK}{GK}:\frac{AB}{AH}$, that is, P : W : : tan ◁ KGI : ſec ◁ AHB.

Corol. 3. If a curve A D H of any kind (having its curvature on each ſide the axis DO exactly alike) be ſuſtained in equilibrio by means of weights B, C, D, *&c.* placed thereon at the points B, C, D, *&c.* indefinitely near each other, any one of thoſe weights, as B, will be directly as the difference of the tangents of the angles, which the radii of curvature drawn from thoſe points, make with the axis of the figure; for, by the foregoing general equation, *viz.* $\overline{w+W}\times\frac{HT}{IT}=W\times\frac{IK}{GK}$, we have w : W : : tan ◁ IHT — tan ◁ GIK : tan ◁ GIK, but



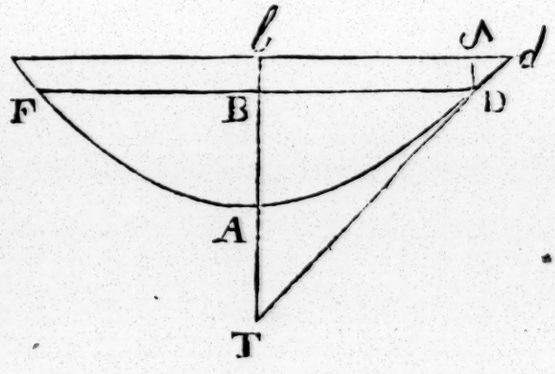
u :

u : W : : tan ◁ HAB — tan ◁ IHT : tan ◁ GIK; therefore if W, and the tangent of GIK are ſuppoſed conſtantly to remain the ſame, then w will be as tan ◁ IHT — tan ◁ GIK, and u will be as tan ◁ HAB — tan ◁ IHT; and conſequently in the above figure, the weight B is equal to the difference of the tangents of the angles BSD, CRD, multiplied by a conſtant (given) quantity.

COROL. 4. The principal property of the catenarian curve may from hence be eaſily derived. Let the given weights be equal to each other, and ſuppoſe $\frac{IK}{GK}$ a conſtant quantity, as it really will be, at the vertex G of the figure (the laſt but one). If now we put l for t, the number of weights or length of the ſtring GIH, a for the conſtant quantity $\frac{IK}{GK} \times W$, $\dot{x}$ for the fluxion of any abſciſſa or variable part of GC, $\dot{y}$ for that of its correſponding ordinate, we ſhall have (by the equation reſulting from the maximum in the problem) $\frac{\dot{y}}{\dot{x}} = \frac{a}{l}$, and conſequently $\dot{y} : \dot{x} : : a : l$, a well known property of the catenaria.

The inveſtigation of this property of the catenaria was, many years ſince, publiſhed in the Philoſophical Tranſactions, by Doctor *David Gregory*, as follows:

" Let FAD be a catena, hanging on the extremities F and D; the loweſt point of which (or the vertex of the curve) is A, the axis AB perpendicular to the horizon, and the ordinate BD parallel to the ſame; we are to find the relation between Bb or Dδ, and $d\delta$, ſuppoſing the point b infinitely near to B, and bd parallel to BD, as alſo Dδ to BA.



From the principles of mechanics, it is plain, that three powers which are in equilibrio, are in proportion to one another as three right lines parallel to their reſpective directions (or inclined in any given angle to them) and terminated at their mutual interſection; and conſequently, if Dd expounds the abſolute gravity of the particle Dd (as it will be in a catena equally thick) then $d\delta$ will repreſent that part of the gravity which acts

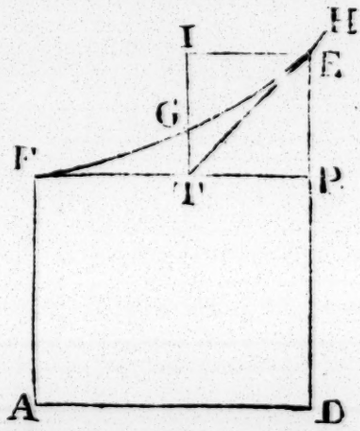
perpendicularly upon D *d*, and by which it comes to paſs, that *d* D (being by the flexibility of the chain moveable about *d*) endeavours to bring itſelf into a vertical poſition. And therefore if ∂ *d* (or the fluxion of the ordinate B D) be conſtant, the action of the gravity exerted perpendicularly upon the correſpondent parts of the catena D *d*, will alſo be conſtant, or every where the ſame. Let this action or force be expounded by *a*.

Farther, from the above cited propoſition in mechanics, D ∂, or the fluxion of the axis AB, will expound the force to be exerted in the direction *d* D, which is equivalent to the former endeavour of D *d* (by which it tends to bring itſelf into a vertical poſition) and is ſufficient to hinder it.

But this force ariſes from the *linea gravis*, DA pulling with the direction *d* D, and is conſequently (all the reſt continuing as before) proportional to that line DA. Therefore ∂ *d*, the fluxion of the ordinate, is to ∂ D, the fluxion of the abſciſſa, as the conſtant quantity *a*, to the curve DA.

If the right line DT touches the catenaria, and meets the axis AB, produced in T; then will DB : BT : : *d* ∂ : ∂ D : :) *a* : DA."

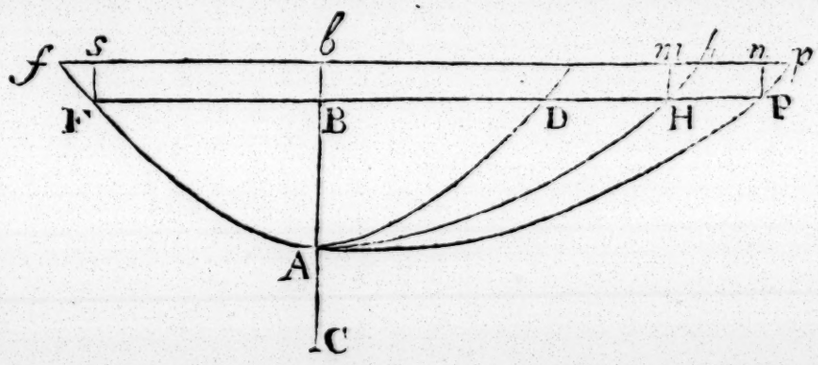
Mr. *Mac Laurin*, at *page* 472 of his Fluxions, gives the following inveſtigation of this property of the catenaria: "When two powers ſuſtain any body or figure that is ſuppoſed to gravitate, a right line from its center of gravity, perpendicular to the horizon, paſſes through the interſection of the right lines in which theſe powers act, which, with the gravity of the figure, are in the ſame ratio to one another, as any three right lines conſtituting a triangle, that are parallel to the reſpective directions of theſe powers. Hence the nature of the figure is diſcovered, which is aſſumed by a heavy chain or perfectly flexible line, that is ſuſpended from any two of its points. Let FEH be ſuch a line, F its loweſtmoſt point, where the tangent FT is parallel to the horizon; ED an ordinate, from E to the horizontal line AD; ET the tangent at E, interſecting FT in T; and G, the center of gravity of the portion E F of the line or chain. Then the three powers are, the gravity of the chain, which acts in the perpendicular to the horizon, and the powers at F and E, which retain thoſe extremities of the chain, by acting in the tangents FT and ET, and are equal to the tenſion of the chain at thoſe points. Therefore, by this principle, the perpendicular from G to the horizon paſſes through T; and if E I, parallel to AD or ET, meet TG in I, the weight of the part of the chain FE, will be to the tenſion of the chain at F, as



I T to E I, or as the fluxion of the ordinate D E, to the fluxion of the baſe A D; conſequently the tangent of the angle I E T, in which the curve interſects a parallel to the horizon at any point E, is always as the weight of the portion F E of the chain at any point E, is to its tenſion at F, as E T to E I (by the ſame principles) or as the fluxion of the curve to the fluxion of the baſe, and is as the ſecant of the angle I E T or E T P."

But to ſhew how this laſt mentioned property, *viz.* That the tenſion at any point E, is as the ſecant of the angle I E T, may be deduced from our method of inveſtigation; let the general equations, $w + W \times \frac{HT}{IT} = W \times \frac{IK}{GK}$, $P \times BH = \overline{w+u+W} \times AH$, and $\overline{u+w+W} \times \frac{AB}{BH} = W \times \frac{IK}{GK}$, (ſee figure on *page* 95) be again reſumed, then by the firſt we have $w+W = \frac{IT}{HT} \times \frac{IK}{GK} \times W$, and by the ſecond $P \times \frac{BH}{IH} = w+W$ (u being $= o$) whence $P \times \frac{TI}{HI} = \frac{IT}{HT} \times \frac{IK}{GK} \times W$; conſequently P is as $\frac{HI}{HT}$ (ſuppoſing $\frac{IK}{GK} \times W$ conſtant) that is, P or the force in I, is as the ſecant of the angle I H T. By the ſame way of reaſoning it will appear, that P, or the tenſion when u takes place, will be as the ſecant of the angle H A B; for $u+w+W = \frac{P \times BH}{AH}$, and therefore $P \times \frac{AB}{AH} = W \times \frac{IK}{GK}$; hence P is as $\frac{AH}{AB}$, or the ſecant of the angle H A B, which ſhews that (in the laſt figure) the tenſion at E, is as the ſecant of the angle I E T or E T P.

Now in order to determine the algebraical equation of the *catenaria*, let an equilateral hyperbola A H, whoſe ſemiaxis $AC = a$, be deſcribed upon the perpendicular A B, as an axis; as alſo upon the ſame axis and vertex, a parabola A P, whoſe parameter is quadruple the axis of the hyperbola; then will B F, (making H F = A P) the ſemiordinate of the *catenaria*, be equal to the difference between the length A P of the curve of the parabola, and the ordinate of the hyperbola; for put A B $= x$, then B b



$= \dot{x}$, and BH $= \sqrt{2ax+x^2}$, whence mh the fluxion of BH $= \frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}}$. Again, ſince the parabola AP has for its parameter $8a$, BP will be equal to $\sqrt{8ax}$, whoſe fluxion up is $\frac{2a\dot{x}}{\sqrt{2ax}}$. Wherefore the fluxion of the curve AP is $\sqrt{\frac{4a^2\dot{x}^2}{2ax}+\dot{x}^2} = \frac{\sqrt{2a\dot{x}^2+x\dot{x}^2}}{x}$, which is equal to $\frac{2a\dot{x}+x\dot{x}}{\sqrt{2ax+x^2}}$; as appears by multiplying both numerator and denominator into $\sqrt{2a+x}$. And ſince HF is every where $=$ AP, the fluxion of HF, that is, $mh+sf$ will be $\frac{2a\dot{x}+x\dot{x}}{\sqrt{2ax+x^2}}$. But we have hitherto found $mh = \frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+x^2}}$, therefore sf (the fluxion of BF) the ordinate in the *catenaria*, $= \frac{a\dot{x}}{\sqrt{2ax+x^2}}$, and conſequently the fluxion of the curve AF (that is, $Ff = \sqrt{sf^2+Fs^2}$ $= \sqrt{\frac{a^2\dot{x}^2}{2ax+x^2}+\dot{x}^2}$) is $\frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+x^2}}$; the fluent of which was ſhewn but now to be $\sqrt{2ax+x^2}$; and therefore AF $= \sqrt{2ax+x^2}$. And it is plain, that the fluxion of the ordinate BF, or $\frac{a\dot{x}}{\sqrt{2ax+x^2}}$, is to $\dot{x}$, the fluxion of the abſciſſa AB, as the conſtant quantity a, to the curve AF, which was the property of the *catenaria* found above. If the ſemiordinate of the *catenaria* be put equal to y, then $\frac{a\dot{x}}{\sqrt{ax+xx}} = \dot{y}$; and by taking the fluent of each ſide the equation, we ſhall have $ca \times \log. \overline{aa+ax+a\sqrt{2ax+xx}} = y$, which, when $x = 0$, becomes $a \times \log. aa$; but then $y = 0$ alſo, therefore $a \times \overline{\log. aa+ax+a\sqrt{2ax+x^2}} - a \times \log. aa$, or $a \times \log. \frac{a+x+\sqrt{2ax+x^2}}{a} = y$. Let z be put for the length of the curve, then $\sqrt{2ax+x^2} = z$, and conſequently $x = \sqrt{zz+aa} - a$.

Thoſe who are deſirous of ſeeing the inveſtigation of other curious properties of the catenarian curve, will find them elegantly done by Doctor *Gregory*, in the ſecond volume of the *Miſcellanea Curioſa*, publiſhed in the year 1708.

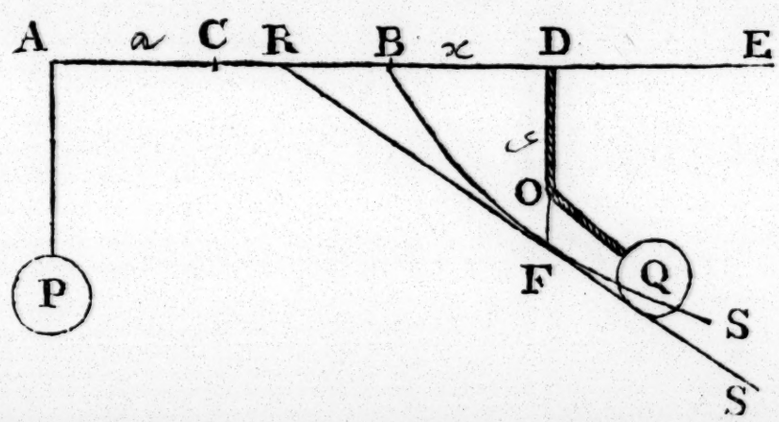
PROBLEM

PROBLEM XLII.

LET C, equidiſtant from the points A, B, be the fulcrum of the indefinite lever ACBE. From the point B, it is required to draw a curve BFS, ſuch that, having appended a given weight P, at the end A of the lever, and drawn from any point in AE, as D, a right line DF perpendicular to AE, and SRF a tangent to the curve in the point F; a weight Q, equal to the former weight P, reſting on the tangent RFS, as an inclined plane, and ſupported in a direction QO, parallel to this tangent, by means of a ſtring QOD, faſtened to the point D, and weight Q, and paſſing under a ſmall pulley at the point O, ſhall be an exact counterpoiſe to the weight P.

SOLUTION.

It is evident, from the nature of the lever, that $\frac{AC}{CD} \times P$, will expreſs the weight, which, being freely ſuſpended from the point D, will keep P in equilibrio. Now if we imagine QOD to ſlide freely over the point D, and to its end a weight appended, which ſhall be juſt ſufficient to retain Q at reſt upon the inclined plane RFS; that weight, and the former which kept P in equilibrio, *viz.* $\frac{AC}{CD} \times P$, will be equal. This premiſed, put $AC = CB = a$, $BD = x$, and $DF = y$; then the ſubtangent DR will be expreſſed by $\frac{y\dot{x}}{\dot{y}}$, therefore $RF = \frac{y}{\dot{y}} \times \sqrt{\dot{x}^2 + \dot{y}^2}$. By Problem III, we have RF : DF : : Q : $\frac{DF}{RF} \times Q$, the force whereby Q endeavours to deſcend along the inclined plane RS; but this force is alſo equal to $\frac{AC}{CD} \times P$, therefore, (becauſe $Q = P$) $\frac{DF}{RF} = \frac{AC}{CD}$, and RF : DF : : CD : AC; that is,



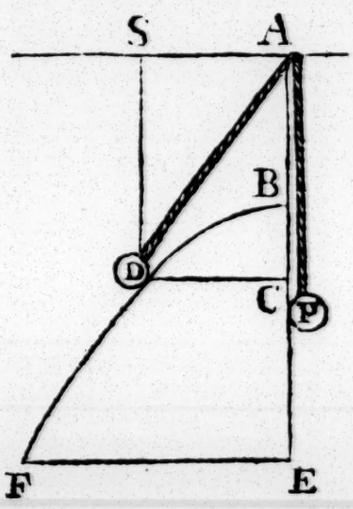
$\frac{y}{\dot{y}} \times \sqrt{\dot{x}^2+\dot{y}^2} : y :: a+x : a$, or $\sqrt{\dot{x}^2+\dot{y}^2} : \dot{y} :: a+x : a$, whence $\dot{x}^2 + \dot{y}^2 : \dot{y}^2 :: \overline{a+x}^2 : a^2$, and $\dot{x}^2+\dot{y}^2-\dot{y}^2\ (\dot{x}^2) : \dot{x}^2+\dot{y}^2 :: \overline{x^2+a^2-a^2}\ (2ax + xx) : \overline{a+x}^2$, conſequently $\dot{x} : \sqrt{\dot{x}^2+\dot{y}^2} :: \sqrt{2ax+xx} : a+x$; multiply extremes and means, we have $a\dot{x}+x\dot{x} = \sqrt{\dot{x}^2+\dot{y}^2} \times \sqrt{2ax+xx}$, whence $\frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+x^2}} = \sqrt{\dot{x}^2+\dot{y}^2}$; and by taking the fluent of each ſide the equation, the length of the curve BF becomes equal to $\sqrt{2ax+xx}$, and therefore the curve BFS is the *catenaria*.

This curious problem I received from the authors of the late Imperial Magazine, to whom it was ſent (in a letter ſigned *George Lane)*; but as that work ſubſided before the receipt of Mr. *Lane*'s Letter, it has obtained a place in this treatiſe. There is a problem ſimilar to this, propoſed by the younger *Bernoulli*, in the *Journal des Scavans*, vol. xx. p. 280; but no ſolution (till the above) was ever given to it, as I can find.

PROBLEM XLIII.

LET the axis EB of the given parabola EBF, be produced to a certain point A; over this point, by means of a ſmall pulley, ſuppoſe a ſtring PAD (to whoſe ends the weights P and D are faſtened) freely paſſes; it is required to find the poſition of the weights D and P, the former reſting upon the curve of the parabola, and the latter freely ſuſpended, when they are a juſt counterpoiſe to each other.

SOLUTION.



Through the point A, draw a right line parallel to the horizon; and let D be the point required, in which D will be ſuſtained at reſt by means of the weight P. Draw DS parallel to the axis BE; and from D, draw the ordinate DC, and join the points D, A. Put $AB = a$, $BC = x$, $CD = y$, $2p$ for the *latus rectum* of the parabola, and l for the length of the ſtring PAD; then by the nature of the curve, we have $2px = yy$, whence $AD = \sqrt{2px+\overline{a+x}^2}$, and $AP = l - \sqrt{2px+\overline{a+x}^2}$, but $AC = x+a$; and

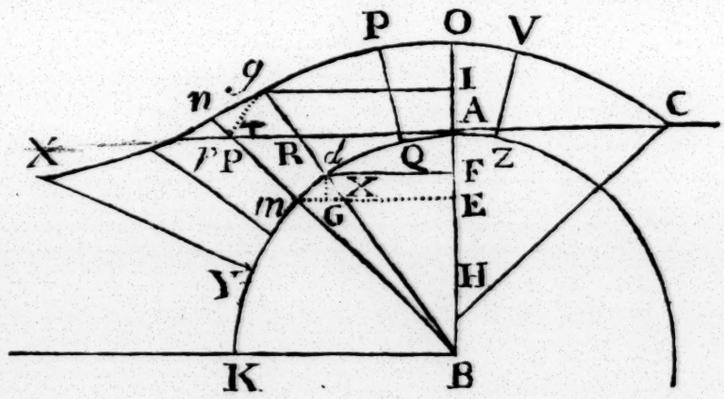
conſequently, by Lemma 1, the perpendicular diſtance of the common center of gravity of the weights D, P, from the horizontal line AS, will be expreſſed by $\frac{P \times l - P \times \sqrt{2px + \overline{a+x}^2} + D \times a + D \times x}{P+D}$, which muſt be the greateſt poſſible. Therefore $D\dot{x} = P \times \frac{p\dot{x} + \overline{a+x} \times \dot{x}}{\sqrt{2px + \overline{a+x}^2}}$, whence $D \times AD = P \times p + P \times AC$.

PROBLEM XLIV.

TO find the nature of the extrados VPX of the propoſed circular arch A*d*Y, ſuch, that the parts thereof ſhall be in equilibrio by the weights of the vouſſoirs alone, without the help of any wall or counterfort, provided only that the abutments be of ſufficient ſtrength to bear the weight of the whole arch.

SOLUTION.

Let B be the center of the circular arch A*d*Y, AB its vertical radius, BK the horizontal one, at right angles to each other; and ſuppoſe *g d m n* (whoſe joints *g d*, *m n*, being produced, would paſs through the center B) one of the vouſſoirs, whoſe extrados *g n* is required. Produce BA to O, ſo that the axis of the key-ſtone QPVZ may be of the given length; draw an indefinite tangent, as CATX, to the point A, the vertex of the arch. From the points *d*, *m*, (ſuppoſed indefinitely near each other) let fall upon AB the perpendiculars *d*F, *m*E; and alſo *d*G perpendicular to *m*E. On B, as center with the radius B*g*, deſcribe the arch *g p*, interſecting B*m n* in *p*; let R and T be the points in which the joints B*g*, B*n*, interſect the horizontal line AT. The angles F*d*B, ARB or ATB, are equal, becauſe of the parallels AR, F*d*, and the indefinitely ſmall arch *d m*. Now if *d*B be taken for radius, BF will be the ſine of the angle B*d*F or BTA, and *d m* the ſine of the angle *d*B*m*; therefore BF is to *d m*, as BR



to RT. Moreover, the angles F*d*G, B*dm*, being right ones, the triangles B F *d*, *m*G *d*, will be ſimilar: Put AB $= a$, BF $= u$, FD $= y$, *d*G $=$ FE $= \dot{u}$, G*m* $= \dot{y}$, B*g* $= s$, and BO $= f$; then F*d* : *d*B : : *d*G *dm*, that is, $y : a :: \dot{u} : \frac{a\dot{u}}{y} = dm$; and again, BF : B*d* : : AB : BR, whence BR $= \frac{a^2}{u}$, alſo BF : *dm* : : BR : TR $= \frac{a^3\dot{u}}{u^2 y}$. It has in Corol. 3, Problem XLI, been proved, that if TR be multiplied by a conſtant quantity *b*, taken at pleaſure, the product will give the weight with which the vouſſoir *dn* muſt be charged, in order to preſerve the equilibrium of the arch. Now as the ſolid part of the arch is compriſed between two vertical planes, it is evident, that the faces of the vouſſoirs, which are the trapezia *gdmn* or *gdmp*, may ſerve to expound the weights of the vouſſoirs, therefore *gdmp* will always be equal to $\frac{a^3 b\dot{u}}{u^2 y}$. But by reaſon of the ſimilar ſectors B*dm*, B*gp*, we have $\overline{Bd}^2 (a) : \overline{Bg}^2 - \overline{Bd}^2 (ss - aa) : :$ B*dm* $\left(\frac{a\dot{u}}{y} \times \frac{a}{2}\right) : dgpm \left(\overline{s^2 - a^2} \times \frac{\dot{u}}{2y}\right)$ hence $\overline{s^2 - a^2} \times \frac{\dot{u}}{2y} = \frac{a^3 b\dot{u}}{y u^2}$, and $ss - aa = \frac{2a^3 b}{u^2}$, from whence we get $s = \frac{a}{u} \times \sqrt{2ab + u^2}$; this, when $u = a$, becomes $ff = 2ab + aa$, conſequently $b = \frac{ff - aa}{2a}$. Let this value of b be ſubſtituted in the equation $s = \frac{a}{u}\sqrt{2ab + u^2}$, it becomes $s = \frac{a}{u} \times \sqrt{f^2 - a^2 + u^2}$; draw the ordinate *g*I to the axis OB, then will BI be a fourth proportional to B*d*, BF, B*g*, and is therefore equal to $\sqrt{f^2 - a^2 + u^2}$. Hence this conſtruction:

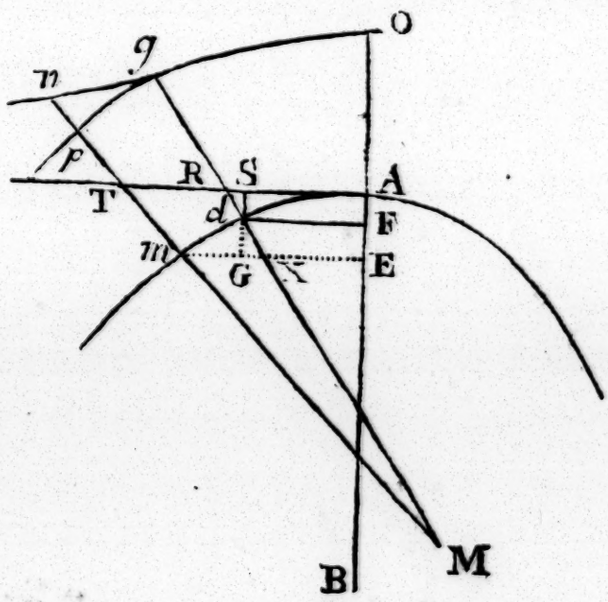
Take the key-ſtone QPVZQ at pleaſure, the lines VZ, PQ, tending to the center B. From A, draw the horizontal line CAT; and from the center B, with BO as radius, deſcribe the circular arch OC, meeting TAC in C; take any point at pleaſure, as *d*, upon the intrados, and make AH equal to the perpendicular diſtance of *d* from the radius BK. Draw CH, make BI $=$ HC; and from I, draw I*g*, meeting B*d* produced in *g*, which will be one point in the required extrados; in the ſame manner other points may be found, by which the extrados O*g*X may be determined.

The

The ſame univerſally.

Let K*d*AZ be an arch of any kind, having its curvature on either ſide the axis AB exactly alike; and from any point therein, as *d*, draw *d*M, the radius of curvature to that point; then other things remaining as before, we ſhall have $m\mathrm{X} = \frac{m\,d^2}{m\mathrm{G}}$, and $\mathrm{R}d = \frac{m\,d \times \mathrm{AF}}{\mathrm{EF}}$ or $\frac{m\,d \times \mathrm{S}d}{d\mathrm{G}}$; therefore $d\mathrm{M} : m\mathrm{X} :: \mathrm{RM} : \mathrm{TR}$, hence $\mathrm{RT} = \frac{m\mathrm{X} \times \mathrm{RM}}{d\mathrm{M}}$. The ſectors M*m d*, M*p g*, are ſimilar, and the area of M*m d* is $\frac{m\,d \times d\mathrm{M}}{2}$; conſequently the area of the trapezium *mp g d* or *mng d*, is equal to $\frac{\overline{\mathrm{M}g^2 - d\mathrm{M}^2} \times m\,d}{2\,d\mathrm{M}}$.

But TR × *b* (*b* being a conſtant quantity) is always equal to *g d mp*, that is, $\frac{\overline{m\,d^2} \times \mathrm{RM}}{m\mathrm{G} \times d\mathrm{M}} \times b = \frac{m\,d \times \overline{\mathrm{M}g^2 - d\mathrm{M}^2}}{2\,d\mathrm{M}}$; divide each ſide of the equation by $\frac{m\,d}{d\mathrm{M}}$, we have $\frac{m\,d \times \mathrm{RM}}{m\mathrm{G}} = \frac{\overline{\mathrm{M}g^2 - d\mathrm{M}^2}}{2\,b}$.



Now to expreſs this equation in algebraic terms, we have AB = *a*, M*g* = *s*, BF = *u*, F*d* = *y*, *d*G = FE = *u̇*, *m*G = *ẏ*; whence the radius of curvature $\mathrm{M}d = \frac{\dot{y}\ddot{y} + \dot{u}\ddot{u} \times \sqrt{\dot{y}^2 + \dot{u}^2}}{-\dot{y}\ddot{u}}$, as is proved by the writers on fluxions. The triangles *m d*G, R*d*S, (G*d* being produced to S) are ſimilar; therefore $-\dot{y} : \sqrt{\dot{u}^2 + \dot{y}^2} :: a - u : \frac{\sqrt{\dot{u}^2 + \dot{y}^2} \times \overline{a-u}}{-\dot{y}} = \mathrm{R}d$; conſequently,

$$RM = \frac{\sqrt{\dot{u}^2+\dot{y}^2}\times\overline{a-u}\times\dot{u}+\overline{\dot{y}\dot{y}+\dot{u}\dot{u}}\times\sqrt{\dot{y}^2+\dot{u}^2}}{-\dot{y}\ddot{u}}, \text{ and}$$

$$\frac{RM\times md}{mG} = \frac{\overline{\overline{a-u}\times\ddot{u}+\dot{y}^2+\dot{u}^2}\times\overline{\dot{y}^2+\dot{u}^2}}{\dot{y}^2\ddot{u}}; \text{ hence } \frac{Mg^2-dM^2}{2b}$$

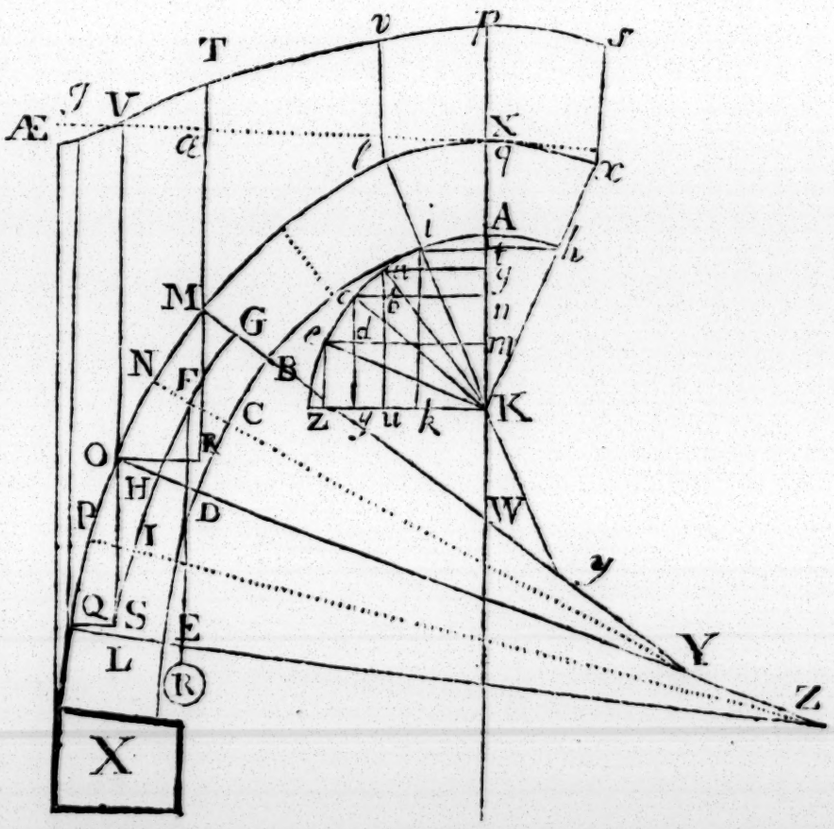
$$\text{or } \frac{ss\times\dot{y}^2\ddot{u}^2-\overline{\dot{y}^2+\dot{u}^2}^3}{2b\times\dot{y}^2\ddot{u}^2} = \frac{\overline{a-u}\times\overline{\dot{y}^2+\dot{u}^2}}{\dot{y}^2\ddot{u}}\times\frac{\dot{u}^2+\overline{\dot{y}^2+\dot{u}^2}^2}{\dot{y}^2\ddot{u}},$$

$$\text{which contracted becomes } \frac{s^2\dot{y}^2\times\ddot{u}}{2b\times\dot{y}^2+\dot{u}^2}-\frac{\overline{\dot{y}^2+\dot{u}^2}^2}{2b} = \overline{a-u}\times\ddot{u}+\dot{y}^2+\dot{u}^2.$$

In this equation, for $\dot{y}$, $\dot{u}$, $\ddot{u}$, ſubſtitute their reſpective values, in terms of y and u, found by the nature of the propoſed arch, there will ariſe an equation free from fluxions, by help of which the figure of the extrados may be determined.

M. *Parent*, in his *Eſſais & Reſcherches de Mathematique*, vol. 3d, gives the following theory of equilibrial arches:

" Imagine the arch AMDQ, divided into ſmall parts by the planes MY, OYZ, QZ, *&c.* perpendiculars to the curvature of the ſaid arch in thoſe points M, N, O, *&c.* indefinitely near each other; by which means the parts MO, OQ, of the arch AMDQ, may be conſidered as right lines. Let MB·DO, ODEQ, be two adjacent vouſſoirs, whoſe common joint produced ODYZ repreſents; let alſo the priſmatic ſolids TMO,



VOQJ, be the charges of thoſe vouſſoirs, neceſſary to keep the parts in equilibrio. We may now conſider theſe ſolids as ſuſtained upon the inclined planes MO, OQ, the upper faces of the vouſſoirs, by means of an adjacent wall or counterfort, which reacts in the horizontal direction ÆX. Theſe inclined planes MO, OQ, are therefore preſſed by the ſolids TMO, VOQ, in the perpendicular directions NY, PZ, in ſuch manner, that the weight of each priſmatic ſolid or charge MO, OV, joined with the reſiſtance of its counterfort, has the ſame effect upon its vouſſoir MD, OE, as if thoſe vouſſoirs were conſtantly acted upon by ſingle forces, acting in the directions NY, PZ; conſequently the vouſſoirs MD, QD, will mutually react with the ſame equality of force in the common direction FI, perpendicular to the center of preſſion H of the joint OD; in the ſame manner the two vouſſoirs OB, OE, will be reacted upon and ſuſtained by the adjacent ones, in the common directions GF, LI, perpendicular to the centers of preſſion G, L, of their joints MB, QE, and will react upon and ſuſtain each in thoſe directions FG, IL, with equality of force; and conſequently the force acting in the direction NY (for example) and the two forces in the directions GF, HF, will be in equilibrio, as will likewiſe the force in the direction PZ, with the two forces acting in the directions HI, LI, and the ſame of all the other vouſſoirs.

Now in order to expound all theſe forces by lines, I draw the horizontal lines OR, QS, meeting the vertical ones TM, VO, produced, in R and S. The three ſides of one of the triangles, as (for example) MRO being perpendiculars to the effort in NY, weight of the charge TO, and the reſiſtance which acts in the direction ÆX; it follows, that MO will expound the firſt effort, RO that of the charge TO, and MR the third effort, and the ſame in the triangle OSQ. The three ſides of the triangle MYO being perpendiculars to the effort in NY, GF and HF, MO will expound the firſt effort, (as in the triangle MOR) MY that acting in the direction GF, and OY the effort in HF. In the triangle OZQ, OZ, OQ, OL will repreſent the reſpective efforts in the directions HI, PZ and LI; therefore the five lines MO, RO, MY, OY, may expound the forces acting againſt the vouſſoir MD, and the five lines OQ, QS, OS, OZ, QZ, may expound thoſe acting againſt the vouſſoir OE. It now remains only to have one line inſtead of the two OV, OZ; and then we may eaſily expreſs, by a range of right lines, the forces acting upon, or the efforts of the vouſſoirs neceſſary to preſerve the equilibrium of the parts. To effect this, produce the ſides *li*, *xh*, of the key-ſtone *xlih*, until they interſect the vertical AK; with AK, the radius of the intrados of the key-ſtone, deſcribe a circular arch, and draw the radii K*a*, K*c*, K*e*, &c. parallel to the joints MY, OVZ, QZ, &c. Draw the

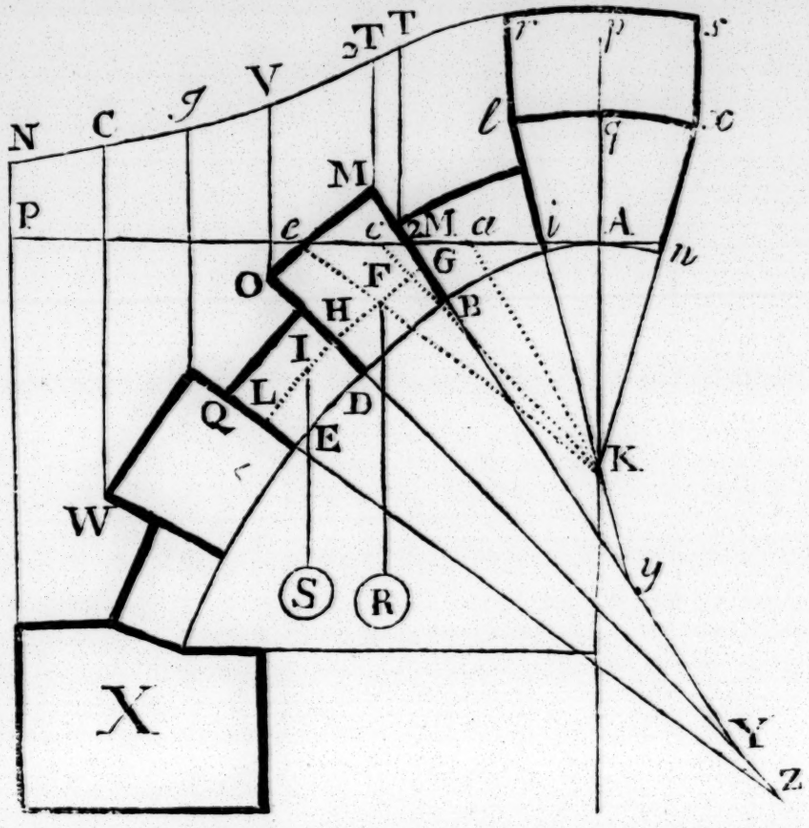
chord *i t h*, draw alſo the right ſines *a g*, *c n*, *e m*, *z* K, and *e* Z, *c y*, *a u*, *i k*. This done, it is evident that the ſectors K *a c*, K *c e*, are ſimilar to the ſectors Y M O, Z O Q, *&c.* and the triangles *a b c*, *c d e*, to the triangles M R O, O S Q; therefore the ſides of the ſector *a* K C, and thoſe of the triangle *a b c*, expound the forces acting againſt the vouſſoir M D, and the ſides of the ſector *c* K *e*, and thoſe of its correſponding triangle *c d e*, do expound the forces acting againſt the vouſſoir O E; but becauſe the radius *c* K ſerves for each vouſſoir M D, O E, it is now manifeſt we may have a range of lines, whereby to expound all the forces of efforts acting upon the vouſſoirs. For the lines *c b*, *e d*, are the differences of the right ſines *a g*, *c n*, *e m*, &c. of the angles *a* K A, *c* K A, *e* K A, which the joints M B, O D, Q E, make with the axis A K of the center; therefore the charger T O, V Q, of the vouſſoirs M D, O E, *&c.* are directly as *c b*, *e d*, &c. and conſequently the entire weights *p q* M T, *p q* O V, *p q* Q J, may be expounded by the right ſines *a g*, *c n*, *e m*, beginning at the axis A K, and taking the ſine *i t*, half the intrados of the key, to expreſs the ſemicharge *p q r*. Hence the nature of the center *x h i l* being given, with the joints M D, O D, Q E, of the vouſſoirs, we may eaſily determine the weights or charges neceſſary to preſerve the parts of the arch in equilibrio. The force acting upon each vouſſoir in the horizontal direction O R, Q S, is expounded by the differences *a b*, *c d* of the coſines *a u*, *c y*, *c z* of the angles *a* K A, *c* K A, which the joints M B, O D, Q E, make with the axis A K; and conſequently if A *t*, the verſed ſine of the angle which *l i* produced makes with the axis A K, be taken to expreſs the force of the ſemicharge *p q l r* of the key, *l i h x*, acting in a direction parallel to the horizon; then A *m*, the verſed ſine of the angle which the laſt joint Q E produced makes with the axis of the figure, will expound the horizontal force of the whole charge of the ſemiarch againſt the wall or counterfort Æ X.

Now as the extrados *x l* of the key-ſtone is to the radius thereof *q* A K, ſo is the charge *s b i r* (the weight of the key-ſtone included) to the horizontal force towards either ſide; hence the figure of the key-ſtone, its weight and charge being given, we may thence eaſily find the two lateral forces.

If we now ſuppoſe the vouſſoirs ſo ſtrongly connected together by means of the maſons work, that the priſmatic ſolids or charges of thoſe vouſſoirs may be ſuſtained without the help of any ſuch wall or counterfort, as was before neceſſary; it is evident, that theſe charges or weights will act upon their correſponding vouſſoirs, in the ſame manner as the weights R, S, (equal to the ſaid charges) ſuſpended from the points F and I.

In

In this caſe, the forces expounded by MR, OS, *&c.* and by MO, QO, or by *a b*, *d c*, and *a c*, *c e*, no longer take place; and conſequently the equilibrium of the vouſſoirs depend only upon the three forces expounded by MY, OY, OR, or by *a* K, *c* K, *c b*, the circular arch *a o e* being now changed into a right A *i a* P, interſected by the right lines K *a*, K *c*, K *e*, drawn parallel to the joints MY, OY, QZ. In the triangle *a* K *c*, the three ſides *a* K, *c* K, *c a*, being perpendiculars to GF, HF, FR, may expound the forces acting in theſe directions; in the ſame manner, the ſides *c* K, *e* K, *c e*, of the triangle *c* K *e*, may be taken to expreſs the forces acting in the directions HI, LI, IS, reſpectively; and becauſe the common ſide *c* K ſerves to expreſs the equal forces in the oppoſite directions HF, HI, it is evident, that the five lines *a* K, *c* K, *a c*, *e* K, *c e*, are as the forces which act upon the two adjacent vouſſoirs MD, OE; and, by proceeding in the ſame manner with the other vouſſoirs, and their correſponding triangles, we ſhall find, that the ſeveral forces with which the adjacent vouſſoirs act upon each other, are as the ſecants *i* K, *a* K, *c* K, *e* K, of the angles which their joints make with the axis *q* K; and that the charges *p l*, *r* 2 M, TO, VQ, *&c.* or the entire weights *p i*, *r* B, TD, VE, *&c.* of the charges and vouſſoirs together, are as the differences A *i*, *i a*, *a c*, *c e*, &c. of the tangents A *i*, A *a*, A *c*, A *e*, &c. of the angles *i* K A, *a* K A, *&c.* which the joints of the vouſſoirs make with the axis *q* K, or, which is the ſame thing, the angles which the radii of curvature make with the axis of the propoſed curve.



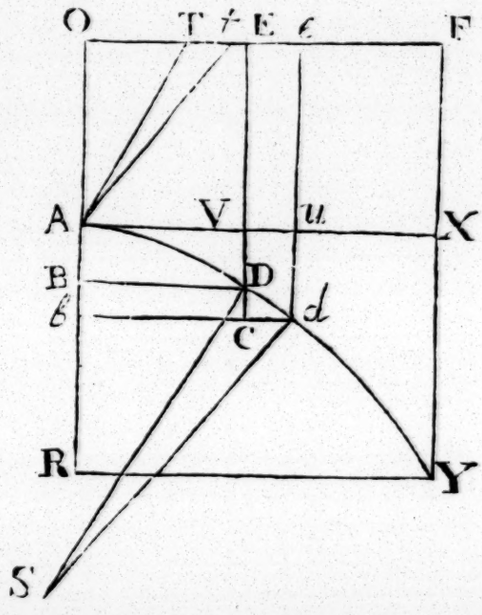
This theory agrees exactly with our determinations in Problem XLI, and its corollaries.

PROBLEM XLV.

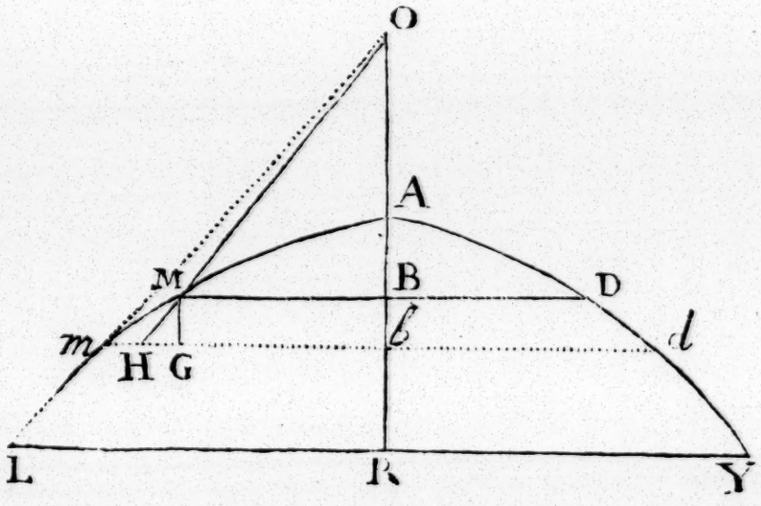
TO determine the nature of the curve ADY, whoſe extrados is terminated by a right line OF, ſuch, that the charges thereon ſhall ſuſtain themſelves in equilibrio by their weights alone, without the help of any wall or counterfort, provided only that the abutments be of ſufficient ſtrength to bear the weight of the whole arch.

SOLUTION.

Let ADY repreſent one half of the required arch, whoſe ſemi-extrados is the right line OF, height AR, ſemibreadth RY, and AO the given height of the wall above the key. From any point B, in the axis AR, draw the ordinate BD; at right angles to AB, draw alſo *b d*, parallel and indefinitely near to BD; to the points D, *d*, draw the radii of curvature DS, *d*S; and from the points D, *d*, draw the verticals DE, *d e*, upon the horizontal line OF. From the vertex A of the curve, draw AT, A*t*, reſpectively parallel to the radii of curvature DS, *d*S. Put AO $= b$, AB $= x$, B*b* $=$ DC $= \dot{x}$, BD $=$ OE $= y$, and C*d* $=$ E*e* $= \dot{y}$. The triangles AOT, *d*CD, are ſimilar, becauſe DC is parallel to AO, and SD to AT; whence C*d* : CD : : AO : OT, that is, $\dot{y} : \dot{x} :: b : \frac{b\dot{x}}{\dot{y}} =$ OT, and conſequently T*t* $= \frac{b\ddot{x}}{\dot{y}}$, ſuppoſing $\dot{y}$ conſtant. Now from the vertex A, draw the horizontal line AX, cutting the verticals ED, *e d*, and YF, in the points V, *u*, and X; then will EV $=$ AO $= b$, VD $=$ AB $= x$, ED $= b + x$; therefore the area of the ſpace E*d*, will be expreſſed by $\overline{b + x} \times \dot{y}$. By Corol. 3, Problem XLI, it appears, that the charge or weight E*d*, neceſſary to preſerve the equilibrium of the part D*d* of the arch ADY, will be equal to T*t*, the difference of the tangents of the angles which the radii of curvature DS, S*d*, make with the axis AR, multiplied by a

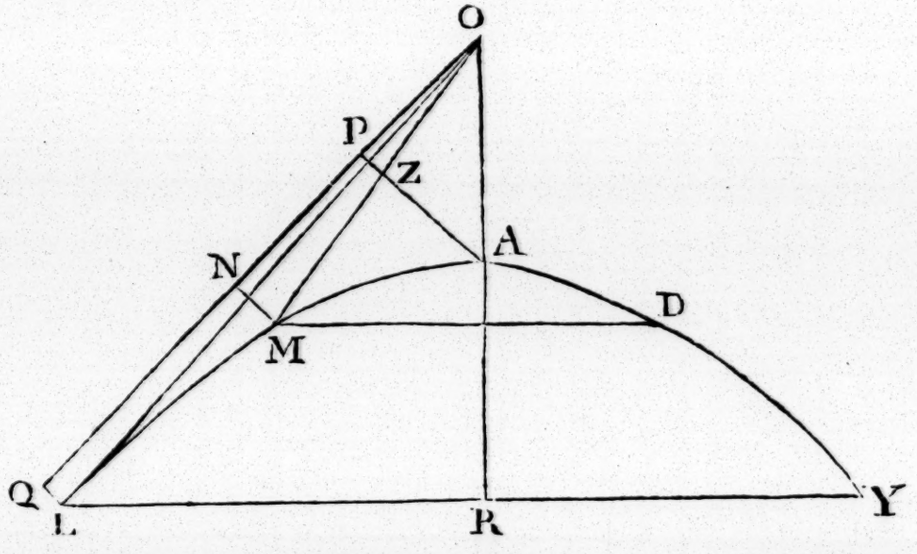


certain conſtant quantity c; hence $\frac{bc\ddot{x}}{\dot{y}} = b\dot{y} + x\dot{y}$, which is the equation of the curve ADY in firſt and ſecond fluxions. Now in order to find the fluent thereof, I multiply each ſide by $2\dot{x}$, which gives $2cb\dot{x}\ddot{x} = 2b\dot{x}\dot{y}\dot{y} + 2x\dot{x}\dot{y}\dot{y}$, whoſe fluent ($\dot{y}$ being conſtant) is $cb\dot{x}\dot{x} = 2bx\dot{y}\dot{y} + xx\dot{y}\dot{y}$; divide the whole by $2bx + xx$, and extract the ſquare root of each ſide the reſulting equation, we ſhall have $\frac{\sqrt{bc} \times \dot{x}}{\sqrt{2bx + xx}} = \dot{y}$; and by taking the fluent of each ſide the equation, we have $\sqrt{bc} \times \log. \overline{b + x + \sqrt{2bx + x^2}} = y$, which, when $x = 0$, becomes $\sqrt{bc} \times \log. b = y$; but then $y = 0$ alſo; therefore the correct fluent is $\sqrt{bc} \times \log. b + x + \sqrt{2bx + x^2} - \sqrt{bc} \times \log. b = y$, or $y = \sqrt{bc} \times \log. \frac{b + x + \sqrt{2bx + x^2}}{b}$. If the fluent of the equation above found, *viz.* $\frac{\sqrt{bc} \times \dot{x}}{\sqrt{2bx + x^2}} = \dot{y}$, be required by a *conſtruction* rather than by *logarithms*; then for $b + x$, write u, and the equation will become $\frac{\sqrt{bc} \times \dot{u}}{\sqrt{u^2 - b^2}} = \dot{y}$; multiply each ſide by $\frac{bb}{2}$, and put h for $\frac{bb}{2\sqrt{bc}}$, it will be farther transformed into $\frac{bb\dot{u}}{2\sqrt{u^2 - b^2}} = h\dot{y}$, which may be thus conſtructed. Upon the axis AR, deſcribe an equilateral hyperbola AML, whoſe vertex is A, and center O. Produce DB, bd, RY, to the points M, m, L; draw OM, Om, and produce the former to H; draw MG parallel to OR, and put BM $= z$, then will G$m = \dot{z}$; and hence OB : BM : : MG : GH, that is,



$u : z :: \dot{u} : \frac{z\dot{u}}{u} = GH$; therefore mH, the baſe of the triangle OHm $= OMm$, will be expreſſed by $\dot{z} - \frac{z\dot{u}}{u}$, or $\frac{u\dot{z} - z\dot{u}}{u}$; which being multiplied by half the perpendicular $OB = \frac{u}{z}$, gives $\frac{u\dot{z} - z\dot{u}}{2}$ for the area of the triangle OMm. By the nature of the hyperbola, $z = \sqrt{u^2 - b^2}$, therefore $\dot{z} = \frac{u\dot{u}}{\sqrt{u^2 - b^2}}$. Let theſe values of z and $\dot{z}$ be ſubſtituted in the expreſſion $\frac{u\dot{z} - z\dot{u}}{2}$, it becomes $\frac{u^2\dot{u}}{2\sqrt{u^2 - b^2}} \times \frac{\dot{u}}{2} \times \sqrt{u^2 - b^2} = u^2 + \frac{\dot{u}\dot{u} - u^2\dot{u} + b^2\dot{u}}{2\sqrt{u^2 - b^2}} = \frac{b^2\dot{u}}{2\sqrt{u^2 - b^2}}$. Therefore the triangle $OMm = b\dot{y} = t$, which gives $\frac{t}{b} = \dot{y}$; or we have the hyperbolic ſpace $OAM = T$, (T being the fluent of t) whence $\frac{T}{b} = y = BD$.

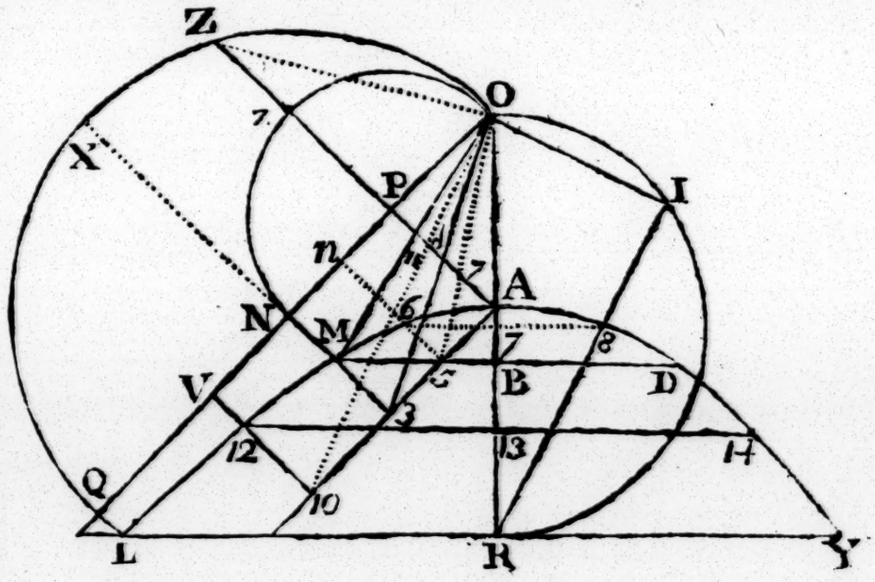
From the center O, draw the aſymptote OQ; and from the points A, M, L, the perpendiculars AP, MN, LQ, to OQ; then, by the property of the hyperbola, $OP \times PA = ON \times NM$; conſequently the triangles OPA, ONM, are equal; ſubtract the common triangle OZP,



there remains the quadrilateral PNMZ, equal to the triangle OZA; and by equal addition of the mixtilineal triangle AZM, we ſhall have the

the aſymptotic complement APNM, equal to the hyperbolic triangle OAM = T. And as this holds good for any point M, it follows, that the aſymptotic complement APQL, is equal to the hyperbolic triangle OAL. Now by finding the area of the ſpace APQL, and having OA, AR given, we ſhall have $\frac{T}{b} = y = RY = f$, and $\frac{T}{f} = b$. In the ſame manner proceed with any other aſymptotic complement APNM = T, and dividing it by *b*, the value of the correſponding ordinate may be found, which (by help of its abſciſſa) determine the points D, *d*, &c. of the required curve.

In order to determine the points D, *d*, &c. of the required curve, by a conſtruction without the help of logarithms, deſcribe on OR the ſemicircle OIR, make OI = OA; draw RI, and make RL = RI. From O, draw an indefinite right line OQ, making with OAR an angle of 45°; from L, let fall LQ perpendicular upon OQ, upon which deſcribe a ſemicircle OZQ. From the vertex A, draw APZ perpendicular to OQ, and A3 parallel to OQ; draw O3, cutting AP in *d*; make ON = OZ, and NM = P*d*. From the point M ſo determined, draw MB perpendicular to the axis AR, which produce, until BD becomes equal to half RY; ſo ſhall D be a point in the required curve ADY.



To find another point, as 8, between A and D, proceed thus. On ON already determined, deſcribe the ſemicircle O*z*N, and make O*n* equal to the chord O*z*. Draw *n*65 perpendicular to OQ; join the points O, 5,

with a right line, cutting A P in 7; make $n6 = P7$; and from the point 6, draw the ordinate 678, making $78 = \frac{RY}{4}$; and 8 will alſo be a point in the required curve A D Y.

In order to determine the points between D and Y, produce 3 N to interſect the periphery of the ſemicircle O Z Q in X. Make O V = O X, and draw V 10 perpendicular to O Q; join the points O, 10, with the right line O 11 10, cutting A P in the point 11; make V 12 = P 11, and from the point 12, draw 12, 13, 14, perpendicular to A R, making 13, 14, equal to $\frac{3RY}{4}$.

In the ſame manner other points in the curve A D Y may be determined; and conſequently, the arch itſelf (of which A D Y is the exact half) may be eaſily deſcribed.

COROLLARY. If, in the preceding equation of the curve, *viz.* $y = \sqrt{bc} \times \log. \frac{b+x+\sqrt{2bx+x^2}}{b}$, the indeterminate quantity c be ſuppoſed equal to b; or, which is the ſame thing, the aſymptotic complement A P Q L equal to half the product of A O into R Y, then will the curve A D Y become the *catenaria*.

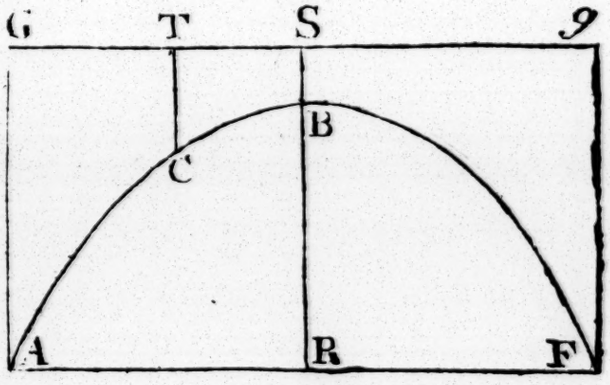
Mr. *Emerſon*, in the ſecond edition of his Mechanics, has given ſeveral methods for conſtructing arches, which ſhall ſuſtain themſelves and all their parts in equilibrio; one in particular this author reckons the moſt uſeful of any, as the arch ſo deſcribed (he informs his readers) will be clear of all thoſe inconveniencies to which the others are ſubject. This arch Mr. *Emerſon* determines in the following manner:

" Make B R, R A, R F, each equal to 30 feet; B S, 3 ½ feet. Draw A G, F *g*, parallel to R S. Divide S G, S *g*, into thirty equal parts, or thirty feet; through all the points of diviſion, draw lines parallel to S R, as T C. Then upon each of theſe lines, ſet off, from S G downwards, the number of feet you find in the following table, reſpectively, as T C; then C will be in the arch. Do the ſame for the ſide S *g*; then the curve F B C A, drawn through all theſe points C, will be the arch required. The curve is eaſily drawn through theſe points, by

help of a bow held to every three points; or rather to four or five points at once, which may be eaſily done by two or three perſons holding it.

Value of ST in feet.	Value of TC in feet and dec. parts.	Value of ST.	Value of TC.	Value of ST.	Value of TC.
0	3.500				
1	3.517	11	5.754	21	14.014
2	3.568	12	6.231	22	15.417
3	3.653	13	6.769	23	16.970
4	3.774	14	7.372	24	18.687
5	3.931	15	8.047	25	20.585
6	4.127	16	8.799	26	22.682
7	4.362	17	9.636	27	24.999
8	4.639	18	10.567	28	27.557
9	4.961	19	11.600	29	30.381
10	5.332	20	12.745	30	33.500

If the thickneſs of the arch at top, B S, be ſuppoſed to be three feet, four feet, or five feet, *&c.* it will require a different curve to be conſtructed; but this ſeems to be ſtrong enough for the bigneſs of the arch, eſpecially if built of ſtrong, ſound ſtone. Here three feet and a half is aſſigned for the thickneſs of the arch; but it muſt be made two or three inches leſs, on account of the parapet wall; for this adds weight to the whole. Alſo if the top G S is not exactly horizontal, but is two or three feet lower at G than at S, the thickneſs B S ought to be two or three inches leſs upon that account; or if higher at G, two or three inches more: but theſe niceties make no ſenſible difference in practice.



This curve differs from the catenary; for at the vertex B, it is leſs curve than the catenary, and towards A it is more curve. The curva-

ture at B in this arch, is very near that of a circle, whoſe radius is B R; and the curvature increaſes from the vertex B, and is leaſt about C.

At the points A, F, where the arch ſprings, it riſes at an angle of 88°. 15′. above the horizon.

If an arch is required to be either greater or leſs than this, it is no more than taking any other equal parts inſtead of feet, and ſetting off all the lines by theſe equal parts."

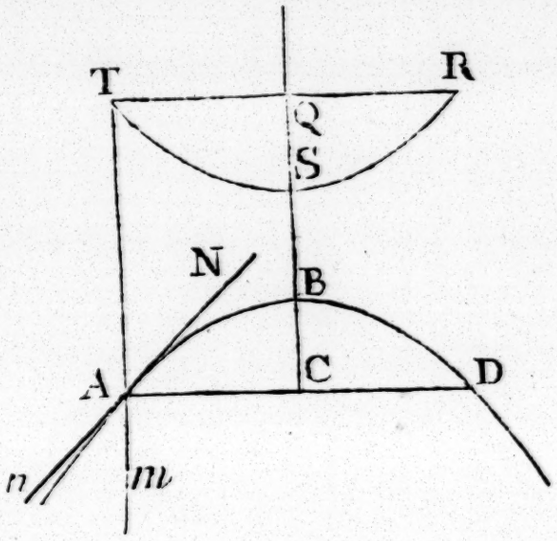
According to the method here laid down by Mr. *Emerſon* for conſtructing an arch of equilibration, it ſhould ſeem as if A F (uſually called the ſpan of the arch) muſt conſtantly be double the perpendicular height R B, and likewiſe equal to thirty feet, yards, *&c.* or thirty ſuch equal parts, of which S B is three and a half. Admitting this, yet we are at a loſs to know whether the equality between A R (= R F) and R B muſt continue, when B S is of any other length than three and a half: becauſe when B S is four feet, or five feet, *&c.* it will, as Mr. *Emerſon* informs us, require a different curve to be conſtructed; and therefore how far the equality of the height and ſemiſpan, which in the former caſe was preſerved, may now become affected by the alteration in the curve, is not eaſily determined; eſpecially as the principles, upon which the foregoing table was conſtructed, do not appear in the abovementioned gentleman's treatiſe.

If the reader is inclined to examine the foregoing table by our theory, it may be done by again reſuming the equation $y = \sqrt{bc} \times \log. \frac{b+x+\sqrt{2bx+x^2}}{b}$; wherein (*c* being indetermined) if we take $b =$ 3. 5. and $y = x = 30$, the value of *c* will become known; and conſequently, by taking the values of T C, as they ſtand in the table, for $b+x$, the correſponding values of *y*, or thoſe of S T, may by the above equation be determined. Or if the values of *y* be aſſumed equal to 1. 2. 3. 4. *&c.* as they ſtand in the table, the correſponding values of *x* may be found; but the former method will be found moſt commodious. The ratio of the S T's may be obtained from the foregoing equation, without firſt determining the value of *c*; for as *y* is there equal to $\log. \frac{b+x+\sqrt{2bx+x^2}}{b} \times \sqrt{bc}$, it is evident that *y* is always as $\log. \frac{b+x+\sqrt{2bx+x^2}}{b}$; therefore,

by

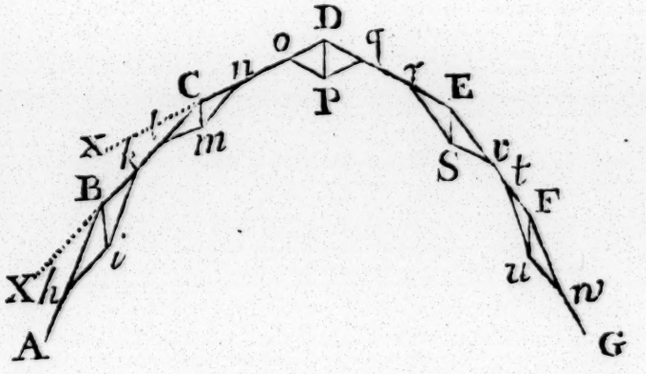
by expounding x by any two (aſſumed) values of TC — BS, the ratio of the correſponding y's or ST's will become known.

The ſame gentleman has, in his Treatiſe on Fluxions, *page* 325, given a theory of arches of equilibration; to which are added, ſeveral examples ſerving to illuſtrate its uſe. Theſe, together with the theory, I ſhall firſt tranſcribe, and then examine by our general method of inveſtigation.



" The nature of the curve ABD, forming an arch, being given; to find the nature of the curve RST, bounding the top of the wall ATRD, ſupported by that arch; by the preſſure or weight of which wall, all the parts of the arch are kept in equilibrio without falling.

1. Let ſeveral equal right lines AB, BC, CD, &c. placed in a vertical plane, moveable round the angles A, B, C, D, &c. whilſt the points A, G, at the baſe, remain fixed and immoveable. Through B, C, D, &c. draw the lines Bi, Cm, DP, &c. perpendicular to the horizon; and complete the parallelogram Bhik, and make Cl = Bk; and complete the parallelogram Clmn. In like manner make Do = Cn or lm, Er = op, Ft = rs; and complete all the parallelograms in the figure as at firſt.



2. Let ſeveral weights, which are to one another as the lines Bi, Cm, Dp, &c. lie reſpectively on the points B, C, D, &c. Now the force Bi, is equivalent to Bh, Bk, acting in the directions BA, BC; the force Bh is deſtroyed by the reſiſtance of the point A; but Bk endeavours to move the point B towards C. In like manner the force CM is equivalent to Cl and Cn; the force Dp to Do, op, &c. Now the forces Bk

(acting towards C) and Cl (acting towards B) being equal by conſtruction, deſtroy one another. In like manner the forces Cn, Do, Dp and Er, Ev and Ft, &c. deſtroy one another; and the point G being fixed, it is manifeſt the figure ABCD, *&c.* will not be moved by the incumbent weights Bi, Cm, Dp, &c. but all its parts will remain in equilibrio.

3. The force Bh : force Bk or Cl : : ſine $<$ hiB or iBC : ſin $<$ ABi $::$ $\frac{1}{\text{ſin} < \text{AB}i}$: $\frac{1}{\text{ſin} < i\text{BC}}$ or $\frac{1}{\text{ſin} < m\text{CB}}$. Likewiſe force C$l$: force Cn or Do : : ſin MCD or pDC : ſin mCB : : $\frac{1}{\text{ſin } m\text{CB}}$: $\frac{1}{\text{ſin } m\text{CD}}$ or $\frac{1}{\text{ſin } p\text{DC}}$, and ſo on; whence it is plain in general, that any force Cl is as $\frac{1}{\text{ſin} < m\text{CB}}$. Now ſince C$m$ $= \frac{\text{ſin C}lm \times \text{C}l}{\text{ſin C}ml} = \frac{\text{ſin BC}x \times \text{C}l}{\text{ſin } m\text{CD}}$; therefore the force C$m$ is as $\frac{\text{ſin BC}x}{\text{ſin } m\text{CB} \times \text{ſin } m\text{CD}}$.

4. Now let the number of the lines AB, BC, CD, *&c.* be increaſed and their lengths diminiſhed *ad infinitum*, that the figure may obtain the form of a curve; and the preſſure will then act on all parts of it, and the angle BCx will then become the angle of contact, and the ſines of mCB and mCD become equal to the ſine of mCx. Therefore the tangent An, (ſee the laſt figure but one) the preſſure on any point A, to preſerve the equilibrium, will be as the angle of contact at A directly, and the ſquare of the ſine of the angle mAn reciprocally. But the angle of contact is as the curvature, or reciprocally as the radius of curvature; therefore the preſſure is reciprocally as that radius, and the ſquare of the ſine of that angle mAn.

5. Let BC $= x$, AC $= y$, AB $= z$, radius of curvature in A $=$ R. Then if z be given, ſine $<$ TAN will be as $\dot{y}$, and $\frac{1}{\text{ſin TAN}}$ as $\frac{1}{\dot{y}}$. Then the weight or preſſure on A $=$ AT $\times$ $\dot{y}$, and that (as has been proved) is as $\frac{1}{\text{R} \times \text{ſin TAN}^2}$ as $\frac{1}{\text{R} \times \dot{y}^2}$; therefore AT is as $\frac{1}{\text{R} \times \dot{y}^3}$; or, in general, AT is as $\frac{\dot{z}^3}{\text{R}\dot{y}^3}$. But when $\dot{x}$ is given, R $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; or if

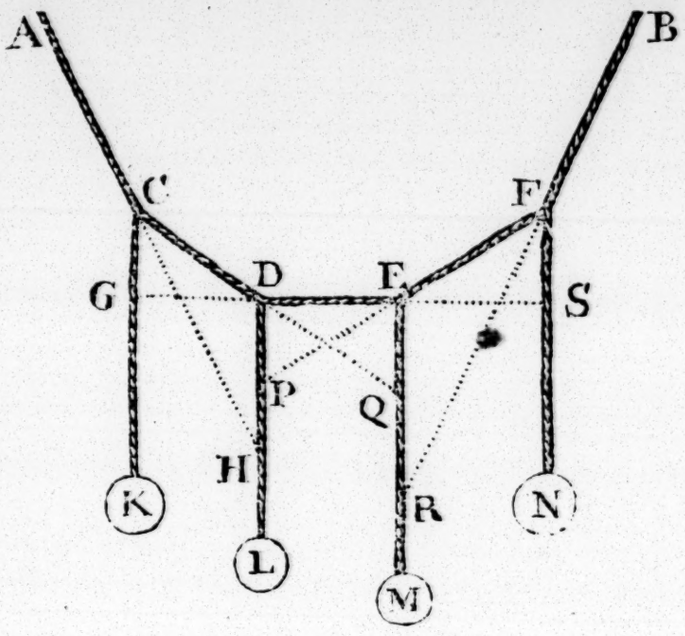
$\dot{y}$ be given, $R = \frac{\dot{z}^3}{\dot{y}\,\ddot{x}}$; therefore AT is as $\frac{-\dot{x}\,\ddot{y}}{\dot{y}^3}$, if $\dot{x}$ be given; or AT as $\frac{\ddot{x}}{\dot{y}^2}$, if $\dot{y}$ be given.

Wherefore to find the curve ST, let $BC = x$, $AC = y$, $AB = z$; then, by the nature of the curve AB, compute $\frac{-\dot{x}\,\ddot{y}}{\dot{y}^3}$, if $\dot{x}$ be given; or $\frac{\ddot{x}}{\dot{y}^2}$, if $\dot{y}$ be given; and take AT proportional thereto. And x may be expunged out of the value of AT, by help of the given line BS; and thence the nature of the curve ST will be known."

The former part of this theory, relating to arches of equilibration, is the ſame with that at *page* 90 of the ſecond edition of his Treatiſe on Mechanics, and may be eaſily ſhewn to agree exactly with the concluſions in the XXXIXth Problem of this work; for it there appears, that $2Q : R$ (ſee the figure to that problem) $:: \frac{DW}{CW} - \frac{ET}{DT} : \frac{ET}{DT}$, and $2P : R :: \frac{SC}{SB} - \frac{DW}{CW} : \frac{ET}{DT}$, whence $P : Q :: \frac{SC}{SB} - \frac{DW}{CW} : \frac{DW}{CW} - \frac{ET}{DT}$. Now becauſe $BC = CD = DE$, it follows, by the principles of plain trigonometry, that the ſine of the angle BCD will be expreſſed by $SC \times CW - BS \times DW$; and that of CDE by $WD \times DT - CW \times ET$, radius being unity. Again, BS, CW, CW and DT, are the reſpective ſines of the angles BCS, SCD, CDW and WDE; therefore, according to Mr. *Emerſon*, $P : Q :: \frac{SC \times CW - BS \times DW}{BS \times CW} : \frac{WD \times DT - CW \times ET}{CW \times DT}$, that is, $P : Q :: \frac{SC}{BS} - \frac{DW}{CW} : \frac{DW}{CW} - \frac{ET}{DT}$; which, as it coincides with the above ratio of $P : Q$, found by the application of our general method, is certainly true.

Mr. *Emerſon*'s method of finding (by the reſolution of forces) the ratio of the weights preſſing upon the angular points of ſuſtaining beams; or, which is the ſame thing, the ratio of weights ſuſtained in equilibrio by means of a ſtring (faſtened to two fixed points) to which thoſe weights

are appended, is not new; for in a poſthumous work of M. *Varignon*, called *Nouvelle Mechanique ou Statique*, publiſhed in 1725, at *page* 181, vol. i. it is there ſhewn, that if any number of weights K, L, M, N, *&c.* be freely ſuſpended from the points C, D, E, F, *&c.* of a flexible cord A C D B between the fixed points A, B, to which the extremities of the cord are faſtened, and the ſides AC, CD, DE, *&c.* of the polygon ACDEFB, into which the cord is formed by the weights K, L, M, N, *&c.* reſpectively produced, until they meet CK, DL, *&c.* the parallel directions of thoſe weights, in the points H, G, S, *&c.* the ratio of K to N will be that of $DH \times EQ \times FS$, to $CG \times DP \times ER$; alſo $L : N :: EQ \times FS : DP \times ER$; $K : M :: DH \times EQ : CG \times DP$.



For by a preceding corollary in *Varignon*'s treatiſe, or by *page* 34 of this, it appears, that

$$\left\{\begin{array}{l} K : L :: DH : CG \\ L : M :: EQ : DP \\ M : N :: FS : ER \end{array}\right. \quad \text{whence} \quad \left\{\begin{array}{l} K = \frac{DH \times L}{CG} \\ L = \frac{EQ \times M}{DP} \\ N = \frac{ER \times M}{FS} \end{array}\right.$$

From the firſt and third of theſe equations, we get $K : N :: \frac{DH \times L}{CG} : \frac{ER \times M}{FS}$; for L, ſubſtitute its value found by the ſecond equation, and we have $K : N :: \frac{DH}{CG} \times \frac{EQ \times M}{DP} : \frac{ER \times M}{FS}$, or $K : N :: DH \times EQ$

$EQ \times FS : CG \times DP \times ER$. By the ſecond and third equations, we have $L : N :: \frac{EQ \times M}{DP} : \frac{ER \times M}{FS}$; whence $L : N :: EQ \times FS : DP \times ER$. Again, by the firſt equation, we have $K = \frac{DH \times L}{CG}$, and by the ſecond we find the value of $M \left(= \frac{DP \times L}{EQ}\right)$; therefore $K : M :: \frac{DH}{CG} : \frac{DP}{EQ} :: DH \times EQ : CG \times EP$; whence the law of continuation becomes manifeſt, let the number of ſuſpended weights be what they will.

But, in order to accommodate the above proportions of the weights to Mr. *Emerſon*'s form of expreſſion, let x be put for the ſine of the angle CGD, to the radius 1; then will $\frac{GD \times x}{CD}$ be the ſine of the angle GCD or CDH. The angles KGD, GDL, are equal, becauſe CK is parallel to DL; and the ſine of the angle CDE will be expreſſed by that of its ſupplement QDE, equal to CDG. Therefore the ſines of the angles ACD, ACK, KCD, CDE and LDE, will be equal to $\frac{DH \times GD \times x}{CH \times CD}$, $\frac{GD \times x}{CH}$, $\frac{GD \times x}{CD}$, $\frac{CG \times x}{CD}$ and x, reſpectively; conſequently, $\frac{\text{ſin} < ACD}{\text{ſin} < ACK \times \text{ſin} < KCD}$, and $\frac{\text{ſin} < CDE}{\text{ſin} < CDL \times \text{ſin} < EDL}$, are reſpectively equal to $\frac{DH}{GD \times x}$ and $\frac{CG}{GD \times x}$; whence, according to Mr. *Emerſon*, $K : L :: \frac{DH}{GD \times x} : \frac{CG}{GD \times x}$, or $K : L :: DH : CG$; which agrees exactly with the ratio of K to L, found by M. *Varignon*'s method. In the ſame manner it may be proved, that the ratios of K to M, L to N, *&c.* which are above ſhewn to be as $DH \times EQ$ to $DP \times CG$, $EQ \times FS$ to $DP \times ER$, *&c.* will alſo be as $\frac{\text{ſin} < ACD}{\text{ſin} < ACK \times \text{ſin} < KCD}$

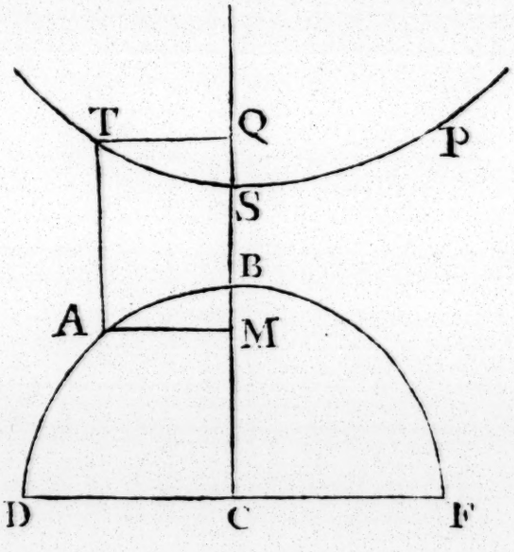
to $\frac{\text{ſin} \sphericalangle DEF}{\text{ſin} \sphericalangle DEM \times \text{ſin} \sphericalangle FEM}$, $\frac{\text{ſin} \sphericalangle CDE}{\text{ſin} \sphericalangle CDL \times \text{ſin} \sphericalangle EDL}$ to $\frac{\text{ſin} \sphericalangle BFE}{\text{ſin} \sphericalangle BFN \times \text{ſin} \sphericalangle EFN}$, *&c.*

PROBLEM XLVI.

TO determine the figure of the extrados of the ſemicircular arch FBAD, ſo that the parts of the ſaid arch ſhall ſuſtain themſelves in equilibrio.

SOLUTION.

Let PST be the required curve of the extrados. Put the radius CD $= CB = r$, the verſed ſine $BM = x$, right ſine $MA = y$, $BS = b$; then



$y = \sqrt{2rx - xx}$, $\dot{y} = \frac{\overline{r-x} \times \dot{x}}{\sqrt{2rx-xx}}$, $\ddot{y} = \frac{-rr\dot{x}^2}{\overline{2rx-xx}^{\frac{3}{2}}}$; then will $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{rr\dot{x}^3}{\overline{2rx-xx}^{\frac{3}{2}}} \times \frac{\overline{2rx-xx}^{\frac{3}{2}}}{\overline{r-x}^3 \times \dot{x}^3} = \frac{rr}{r-x}$; therefore AT is as $\frac{rr}{\overline{r-x}^3}$, or $AT = \frac{r^3 b}{\overline{r-x}^3}$; and expunging x, $SQ = (AT - MS) = \frac{r^3 b}{\overline{rr-yy}^{\frac{1}{2}}} - a - r + \sqrt{r^2 - y^2}$ for the nature of the curve PST.

Otherwiſe.

It appears by the ſolution to Problem XLV, that OT (ſee the figure to that problem) $= \frac{b\dot{x}}{\dot{y}}$, the fluxion of which is $\frac{-b\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, ſuppoſing $\dot{x}$ con-

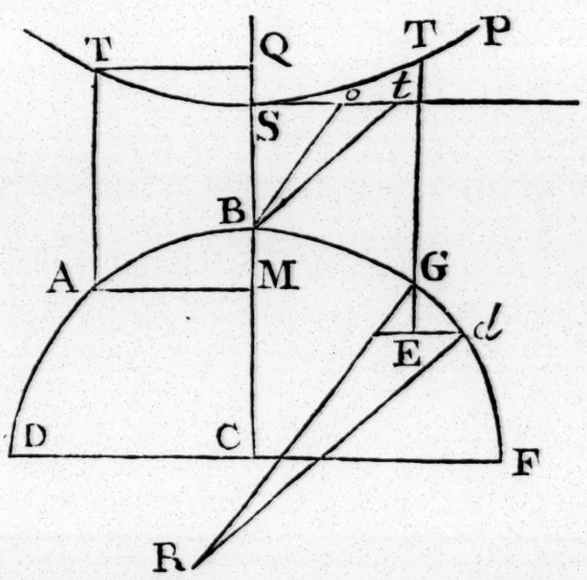
ſtant; therefore by Corol. 3, Problem XLI, $\frac{-b\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, × by a conſtant quantity, is equal AT (in the above figure) multiplied into $\dot{y}$; whence AT $= \frac{-bc\dot{x}\ddot{y}}{\dot{y}^3}$, c being put for the conſtant quantity. Now, by the property of the circle, we have $2rx - xx = yy$, and therefore $-\frac{\dot{x}\ddot{y}}{\dot{y}^3} = \frac{rr}{\overline{r-x}^3}$, whence $AT = \frac{rrbc}{\overline{r-x}^3}$; conſequently $SQ = \frac{rrbc}{\overline{r-x}^2} - b - x$; wherein, if we make $c = r$, (becauſe when $x = o$, SQ muſt vaniſh) and expunge x, we ſhall have $SQ = \frac{r^3 b}{\overline{r^2-y^2}^{\frac{3}{2}}} - a - r + \sqrt{r^2-y^2}$, the ſame as before.

PROBLEM XLVII.

TO determine the figure of the extrados of the ſemielliptical arch FBD, ſo that the parts of the ſaid arch ſhall ſuſtain themſelves in equilibrio.

SOLUTION.

Let PST be the required curve of the extrados. Put the ſemitranſverſe diameter DC $= t$; ſemiconjugate CB $= c$, BM $= x$, MA $= y$; then, by the nature of the ellipſis, we have $y = \frac{t}{c}\sqrt{2cx-xx}$, $\dot{y} = \frac{t\dot{x}}{c} \times \frac{c-x}{\sqrt{2cx-xx}}$, $\ddot{y} = \frac{-tc\dot{x}^2}{\overline{2cx-2x}^{\frac{3}{2}}}$,



ſuppoſing $\dot{x}$ conſtant; whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{c^4}{t^2 \times \overline{c-x}^3}$, therefore $AT = \frac{t^3 b}{\overline{c-x}^3}$.

Then $SQ = \frac{c^3 b}{\overline{c-x}^3} - b - x = \frac{t^3 b}{\overline{t^2-y^2}^{\frac{3}{2}}} - b - c + \frac{c}{t}\sqrt{t^2-y^2}$, by exterminating x; whence the nature of the curve PST becomes known.

Otherwiſe.

From the points G, d, indefinitely near each other, draw the radii of curvature GR, dR; and from the vertex B of the arch, draw Bo, Bt, reſpectively parallel to GR, dR, meeting the tangent St, drawn to the extrados PST, in the points o, t. Draw alſo dE parallel to FD, and GE parallel to BC, which produce to T; then, by the ſimilar triangles GEd, SBo, we have Ed : EG :: BS : So, that is, $\dot{y} : \dot{x} :: b : \frac{b\dot{x}}{\dot{y}} =$ So, whoſe fluxion, making $\dot{y}$ conſtant, is $\frac{b\dot{y}\ddot{x}}{\dot{y}^2} = ot$; therefore by Corol. 3, Problem XLI, $\frac{bd\ddot{x}}{\dot{y}^2} =$ GT, (d being put for the conſtant quantity.) Now, by the nature of the ellipſis, we have $y = \frac{t}{c}\sqrt{2cx-xx}$, therefore $\dot{y} = \frac{t\dot{x}}{c} \times \frac{c-x}{\sqrt{2cx-xx}}$; and if $\dot{y}$ be ſuppoſed conſtant, we ſhall have $\ddot{x} = \frac{c^2\dot{x}^2}{\overline{c-x} \times \overline{2cx-xx}}$, but $\dot{y}^2 = \frac{tt\dot{x}^2}{cc} \times \frac{\overline{c-x}^2}{2cx-xx}$; whence $\frac{bd\ddot{x}}{\dot{y}^2}$ becomes, by ſubſtituting of $\ddot{x}$ and $\dot{y}^2$ their reſpective equals, $\frac{bdc^4}{\overline{c-x}^3 \times tt}$. This expreſſion, when $x = 0$, muſt be equal to b; therefore d muſt be taken equal to $\frac{tt}{c}$, and conſequently AT becomes $\frac{c^3 b}{\overline{c-x}^3}$, the ſame as before.

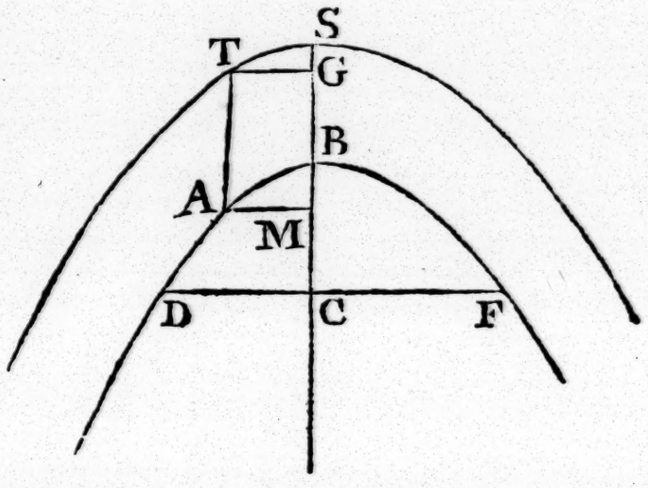
PROBLEM

PROBLEM XLVIII.

TO determine the ſame things as in the two laſt problems, the propoſed arch being the curve of the common parabola.

SOLUTION.

Let the abſciſſa BC be put $= x$, correſponding ſemiordinate CA $= y$; then, by the property of the parabola, we ſhall have $2px = yy$, $2p$ being put for the parameter to the axis. Then $\dot{x} = \frac{2y\dot{y}}{p}$, $\ddot{x} = \frac{\dot{y}\dot{y}}{p}$ ($\dot{y}$ being conſtant); and therefore AT (drawn parallel to the axis BC) which is expreſſed by $\frac{\ddot{x}}{\dot{y}^2}$, when $\dot{y}$ is conſtant, now becomes $\frac{1}{p}$, a conſtant quantity; therefore AT is every where the ſame, and the extrados of the propoſed arch AB is the aſymptotic parabola ST.



PROBLEM XLIX.

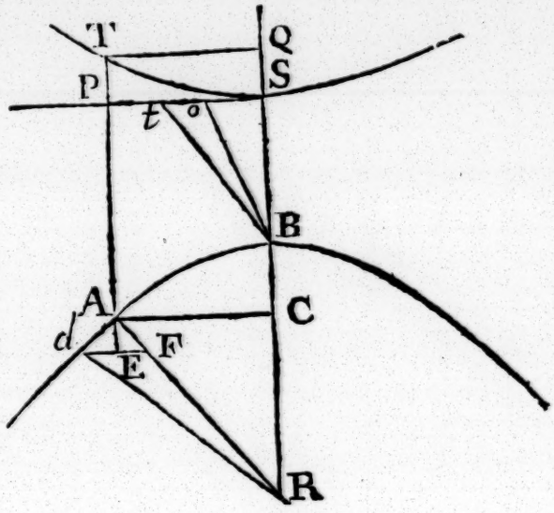
LET BA be the curve of an hyperbola, to determine the extrados thereof ſuch, that the parts of the propoſed arch ſhall ſuſtain themſelves in equilibrio.

SOLUTION.

Put the tranſverſe diameter $= 2r$; conjugate $= 2c$, BC $= x$, CA $= y$, BS $= b$; then, by the property of the curve, $2rx + xx = \frac{rryy}{cc}$,

whence $r+x=\frac{r}{c}\sqrt{cc+yy}$, $\dot{x}=\frac{r\dot{y}}{c}\times\frac{y}{\sqrt{cc+yy}}$, $\ddot{x}=\frac{rc\dot{y}^2}{cc+yy^{\frac{3}{2}}}$;

therefore the general expreſſion for AT, *viz.* $\frac{\ddot{x}}{\dot{y}^2}$, in this caſe becomes $\frac{rc}{cc+yy^{\frac{3}{2}}}$, and $AT=\frac{c^3a}{cc+yy^{\frac{3}{2}}}$.



Hence the curve ST continually approaches the hyperbola; and $SQ=(a+x-AT=)\ a-r+\frac{r}{c}\sqrt{cc+yy}-\frac{c^3a}{cc+yy^{\frac{3}{2}}}$, expreſſes the nature of the curve ST.

Otherwiſe.

Draw the radii of curvature RA, R*d*, indefinitely near each other; draw *d*EF parallel to the ſemiordinate AC, and AE parallel to the axis BC. From the vertex B, draw B*o*, B*t*, reſpectively parallel to radii RA, R*d*, interſecting SP (a tangent to the curve ST in S) in the points *o*, *t*; then will the triangle A*d*E be ſimilar to the triangle BS*o*, therefore E*d*: EA :: BS : S*o*, that is, $\dot{y}:\dot{x}::a:\frac{a\dot{x}}{\dot{y}}=So$; and conſequently $ot=\frac{a\ddot{x}}{\dot{y}}$, ſuppoſing $\dot{y}$ conſtant. By Corol. 3, Problem XLI, it appears, that $\dot{y}$AT, the charge or weight acting upon A*d*, to preſerve the equilibrium in that part, is equal to $\frac{da\ddot{x}}{\dot{y}}$, (*d* being the neceſſary conſtant quantity) therefore $AT=\frac{da\ddot{x}}{\dot{y}^2}$. Now, by the equation of the curve, we have

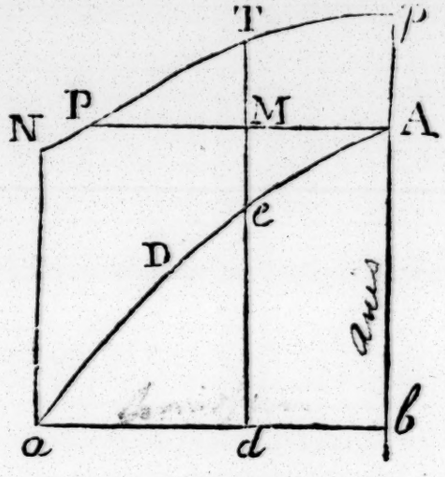
$2rx+xx=\frac{rryy}{cc}$, whence $\dot{x}=\frac{r\dot{y}}{c}\times\frac{y}{\sqrt{cc+yy}}$, $\ddot{x}=\frac{rc\dot{y}^2}{cc+yy^{\frac{3}{2}}}$, and AT $=\frac{rdac}{cc+yy^{\frac{3}{2}}}$. From this value of AT, ſubtract SC, $(a+x)$ and we have $SQ=\frac{rdac}{cc+yy^{\frac{3}{2}}}-a-x$, which, when $y=0$, muſt (together with x) entirely vaniſh; conſequently in that circumſtance, $c^2=rda$, whence $d=\frac{c^2}{ra}$. Subſtitute this value of d, in the expreſſion for AT, it becomes $\frac{c^3a}{cc+yy^{\frac{3}{2}}}$, the ſame as before; and, putting $SQ=z$, the equation of the curve ST will be defined by the equation $z=a-r+\frac{r}{c}\sqrt{cc+yy}-\frac{c^3a}{cc+yy^{\frac{1}{2}}}$.

The former ſolutions in this and the three preceding problems, are the ſame with thoſe given by Mr. *Emerſon*, in his Treatiſe on Fluxions; and exactly coincide with thoſe we have deduced from the theory inveſtigated in the XLIſt Problem and its corollaries, by the application of our general method of reſolving mechanic problems.

By help of theſe inveſtigations, we eaſily arrive at thoſe concluſions which M. *Parent* has given, at *page* 168 of his *Eſſais & Reſcherches de Mathematique*, vol. 3d, namely, That if AD be a given arch, whoſe extrados is *p*TN, axis A*b*, ſemiſpan *ba*, and A*p* the given height of the vertex of the required extrados from that of the arch; and from A, a right line AMP be drawn parallel to the horizontal line *ba*; alſo through any point M, a right line parallel to A*b*; then will T*e*, (A*ba* being a quadrant of a circle) the height of the extrados above the point *e* of the propoſed arch, be inverſely as the cube of the right line *ed*. But if A*ea* be the arch of an ellipſis, then having firſt multiplied the ſquare of *ed*, by the ſquare of the ſemitranſverſe *ba*, and the ſquare of the horizontal diſtance AM by the ſemiconjugate A*b*, and ſubtracted one product from

the other, the height *e* T of the required extrados will be directly as that difference, and inverſely as the cube of *e d*.

If the propoſed arch be the curve of an hyperbola; then, having multiplied the ſquare of the *de*'s, by that of the ſemi-ordinate *b a*, and the ſquare of the horizontal diſtances A M, by the correſponding part A *b* of the axis, and taken the ſum of thoſe products, the ratio of the heights M T will be compounded of the direct ratio of thoſe ſums, and inverſe ratio of the cubes of the heights *d e*.

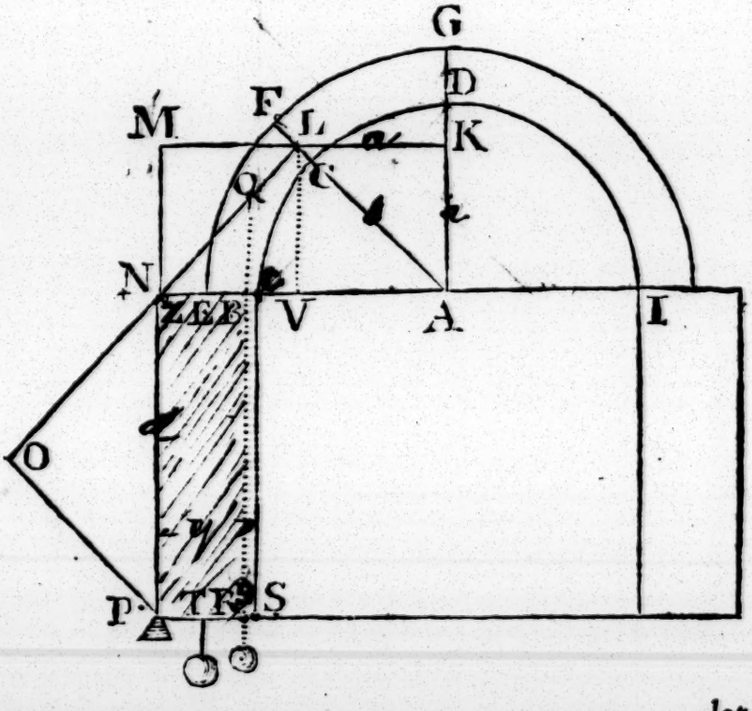


PROBLEM L.

TO determine the thickneſs of the piers neceſſary to ſuſtain in equilibrio the parts of a propoſed ſemicircular arch.

SOLUTION.

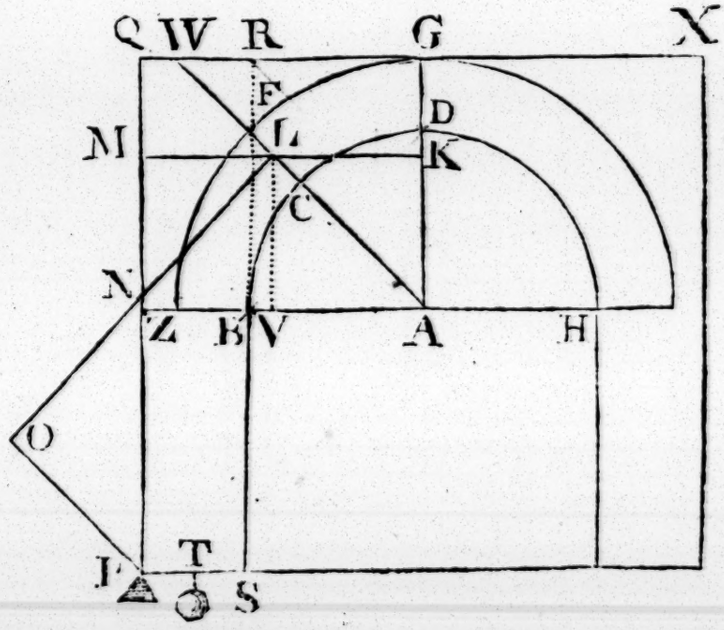
Let S Z be one of the required piers of the ſemicircular arch, whoſe half is repreſented by B E G D: Through L, the middle of F C, draw M K parallel to Z A; and produce P Z to M. Let fall L V perpendicular upon A B; and from L, draw L O at right angles to L A, meeting the perpendicular P O in O. From Q, the center of gravity of the vouſſoir C E,



let

let fall QR perpendicular upon the baſe SP of the pier; biſect PS in T; call LK or KA, a, LA, b, BV, c, ZP, d, ZB or RS, g, PS, y; then will ML or MN be equal to $c+y$, and MP $=a+d$; conſequently NP $=a+d-c-y$. Put $a+d-c=f$, then NP $=f-y$, and PR $=y-g$. If for the ſurface of the vouſſoir CG, or its equal CE, we put nn, then will $\frac{bn^2}{a}$ expreſs the force of the vouſſoir in the direction LO. By the ſimilar triangles LKA, NOP, we have LA (b) : LK (a) : : NP ($f-y$) : PO equal to $\frac{af-ay}{b}$; which, being multiplied by $\frac{bn^2}{a}$, gives n^2f-n^2y for the force acting at right angles upon the arm PO of the lever RPO, whoſe center of motion is P. Now, by Problem XLI, it appears, that $n^2\times\overline{y-g}$ and $\frac{dy^2}{2}$ are the reſpective forces acting at the points T, R, in a direction perpendicular to the horizon, therefore by the nature of the lever (ſee *page* 76) we have $nnf-nny=\frac{ddy^2}{2}+nny-nng$. From this equation, by compleating the ſquare and proper reduction, we get $y=\sqrt{\frac{2n^2f+2n^2g}{d}+\frac{4n^4}{dd}}-\frac{2n^2}{d}$.

When the extrados of the arch BDH is terminated by a horizontal line QX; then having taken FC $=$ DG, and drawn LO perpendicular to FC, in the middle point L, as before, and other things remaining ſtill the ſame, put MP $=f$; then by the ſimilar triangles LKA, PON, we have LA : LK : : PN : PO, whence PO



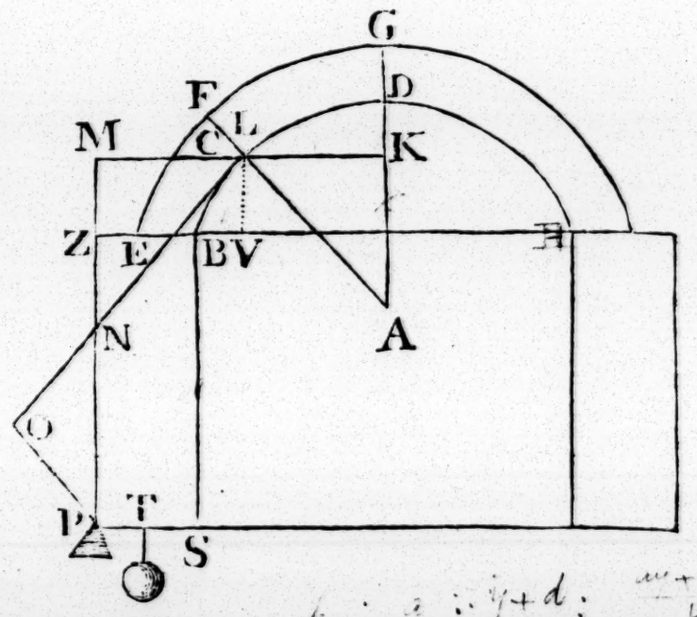
$= \frac{af - ae - ay}{b}$. Now the abſolute weight of the vouſſoir CWGD, here ſuppoſed $= n^2$, is to the force thereof, acting in the direction LO, as LK : LA (Problem III.) and is therefore equal to $\frac{n^2 b}{a}$; which, being multiplied by PO, becomes $fnn - cnn - nny$. And this product muſt, by the nature of the lever, be equal to the weight of the pier PQRS, multiplied by PT; hence this equation, $fn^2 - cn^2 - n^2 y = \frac{dyy}{2}$; from whence we get $y = \frac{n^4}{d^2} - \frac{n^2}{d} + \frac{\sqrt{2fn^2 - 2cn^2}}{d}$.

PROBLEM LI.

TO determine the thickneſs of the piers neceſſary to ſuſtain in equilibrio the parts of a propoſed ſemielliptical arch.

SOLUTION.

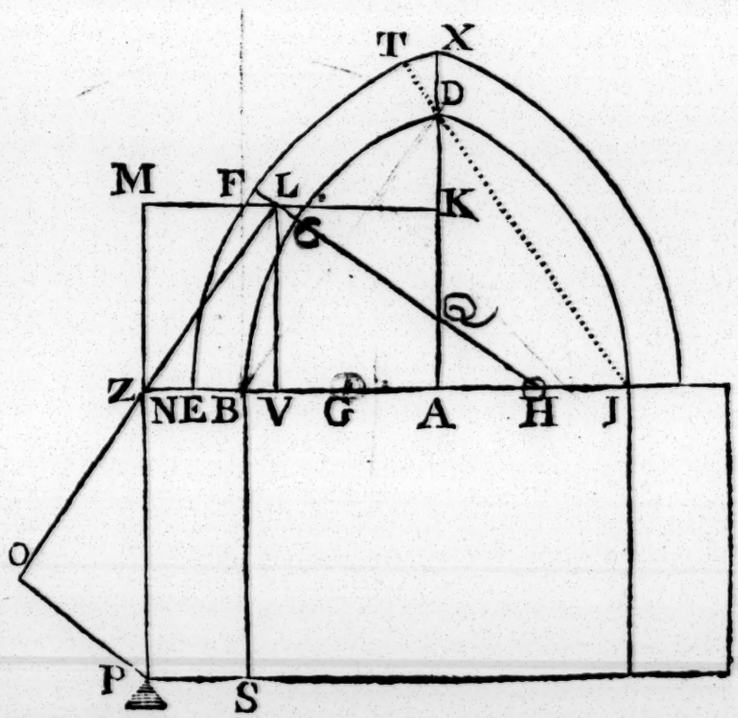
Let SZ be one of the required piers of the ſemielliptical arch, whoſe half is repreſented by BEGD. From the middle of the arch BD, draw LO a tangent to that point; and erect the perpendicular LA, which produce to A. From P, let fall PO perpendicular upon LO; and through L, draw MLK parallel to the ſemitranſverſe axis BH. Put LK $= a$, KA $= b$, LA $= c$, BV $= d$, BS $= f$, MP $= g$, n^2 for the ſurface of the vouſſoir CG, nearly equal to FB, ZB $= y$.



By the ſimilar triangles LKA, LMN, we have AK : LK : : LM : MN;

whence $MN = \frac{ay+ad}{b}$, and thence $NP = \frac{gb-ad-ay}{b}$; and again, by the ſimilar triangles LKO, NOP, we get $PO = \frac{gb-ad-ay}{c}$. Now the abſolute weight of the vouſſoir LFGD, is to its preſſure upon FL, as LK to LA; conſequently $\frac{cn^2}{a} \times \frac{gb-ad-ay}{c}$, expreſſes the force acting upon the point O, in the direction LO. On the other hand, the reſiſtance of the pier SZ will be expreſſed by $\frac{fyy}{2} + ny$ (PT being equal to TS, and P conſidered as the center of motion of the lever OPT); hence, by making the force and reſiſtance equal, we ſhall have $\frac{gbn^2}{a} - dn^2 - n^2y = \frac{fyy}{2} + n^2y$, and conſequently $y = \sqrt{\frac{2gbn^2}{af} - \frac{2dnn}{f} + \frac{4n^4}{f^2}} - \frac{2n^2}{f}$.

If the propoſed arch be of the Gothic kind, formed by the equal parts BD, DI, of two equal circles, whoſe centers are at the points H and G, equidiſtant from A, the middle of BI; then having biſected BD in C, and drawn HCF, HDT; and from L, the perpendicular LO; likewiſe MLK, LV, MP, PO, *&c.* as before; put LK $= a$, KQ $= b$, LQ $= c$, BV $= d$, ZP $= f$, MP $= g$, ZB $= y$. The triangles LKQ, LMN, are ſimilar; hence KQ (b) KL (a) :: LM



$(d+y)$: MN $\left(\frac{af+ay}{b}\right)$; and therefore NP $= \frac{bg-ad-ay}{b}$. Again, the triangle LKQ is ſimilar to the triangle NOP; conſequently LQ (c) : KQ (b) : : NP $\left(\frac{bg-ad-ay}{b}\right)$: PO $\left(\frac{bg-ad-ay}{c}\right)$. Now let the abſolute weight of the vouſſoir LDT be expounded by LK; then will PO $\times \frac{cn^2}{a}$ (n^2 being put for the ſurface of LDT) or $\frac{bgn^2}{a} - dn^2 - n^2y$, expreſs the force acting upon the point P, in the direction LO. On the other hand, we have $\frac{fyy}{2} + n^2y$ for the force acting upon the point P, in the direction BS; therefore, by the property of the lever, theſe forces muſt be equal. Hence $\frac{bgn^2}{a} - dnn - nny = \frac{fyy}{2} + nny$; this equation being properly reduced, gives $y = \sqrt{\frac{2bgn^2}{af} - \frac{2dn^2}{f} + \frac{4n^4}{f^2}} - \frac{2n^2}{f}$.

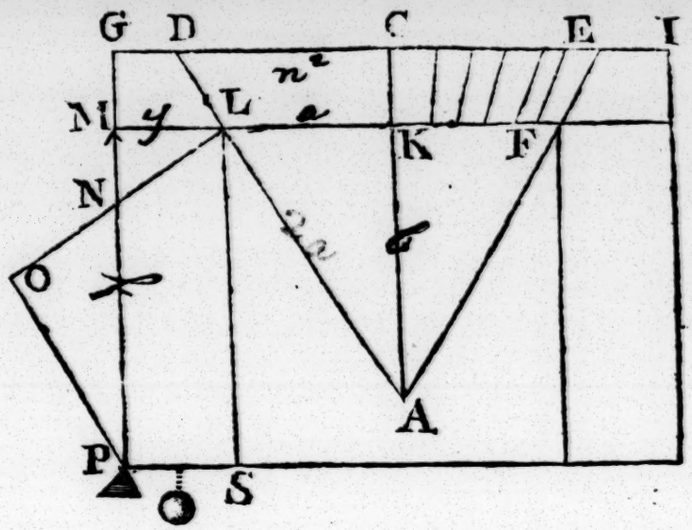
PROBLEM LII.

TO determine the thickneſs of the piers neceſſary to ſupport in equilibrio the propoſed vouſſoirs, when ranged in a right line.

SOLUTION.

Upon LF, the given diſtance between the piers, deſcribe an equilateral triangle LAF; and divide its baſe LF, into as many equal parts as there are vouſſoirs. From A, draw through thoſe points of diviſion, right lines, terminating in the indefinite right line GI (drawn parallel to LF, at a given diſtance therefrom) in the points E, C, D, *&c.* From L, draw LO at right angles to AD; and from P, let fall PO perpendicular thereon. Produce KL to M, and draw AKC perpendicular

perpendicular to LF. Put KL = a, KA = b, MP = f, LM = y, n^2 for the area of LDCK. The triangles AKL, LMN, and NOP, are similar; hence KA (b) : KL (a) :: LM (y) : MN $\left(\frac{ay}{b}\right)$, and consequently NP = $\frac{fb-ay}{b}$. Again, AL ($2a$) : AK (b) :: NP $\left(\frac{fb-ay}{b}\right)$: PO $\frac{fb-ay}{2a}$. Now by Problem III. it appears, that the absolute weight of LDCK is to its force, upon the pier SM, as LK to LA; but LA is double to LK. Therefore the force acting upon O, in the direction LO, may be expounded by $2nn$, which, being multiplied by PO, gives $\frac{bfn^2}{a} - n^2y$, for the force acting upon P, at the distance PO; which, by the property of the lever, must be equal to the force acting upon the point P, at the distance of half PS therefrom. But this is expounded by $\frac{fyy}{2}$; hence $\frac{bfnn}{a} - nny = \frac{fyy}{2}$, and consequently $y = \sqrt{\frac{2bnn}{a} + \frac{n^4}{f^2}} - \frac{n^2}{f}$.



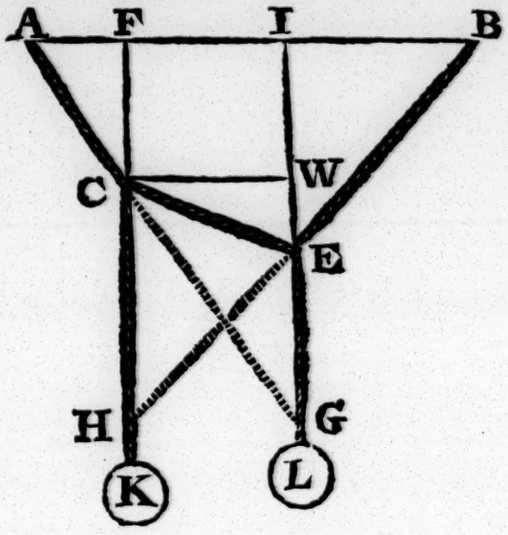
PROBLEM LIII.

LET the string ACEB, whose ends A and B are fastened to two tacks in the horizontal line AB, be divided into the given parts AC, CE, EB, and at the points of division C and E, the known weights K, L, appended; it is required to find the position of the said weights and string, when sustained in equilibrio.

M m

SOLUTION.

From the points C, E, draw CF, EI, reſpectively perpendicular to the horizontal line AB; draw alſo CW parallel to AB. Put $AC = a$, $CE = b$, $EB = d$, $AB = n$, $AF = x$, $BI = y$, then will $FI = CW = n - x - y$, $FC = \sqrt{a^2 - x^2}$, $EI = \sqrt{d^2 - y^2}$, conſequently $\sqrt{bb - \overline{n-x-y}^2} = \sqrt{d^2 - y^2} - \sqrt{a^2 - x^2}$. By Lemma 1, we have $\frac{K \times \sqrt{a^2 - x^2} + L \times \sqrt{d^2 - y^2}}{K + L}$ for the perpendicular diſtance of the common center of gravity of the two weights K, L, from the horizontal line AB, which muſt be a maximum. In fluxions



we have $K \times \frac{-x\dot{x}}{\sqrt{a^2 - x^2}} + L \times \frac{-y\dot{y}}{\sqrt{d^2 - y^2}} = 0$; and by taking the fluxion of the above equation, *viz.* $\sqrt{b^2 - \overline{n-x-y}^2} = \sqrt{d^2 - y^2} - \sqrt{a^2 - x^2}$, we ſhall have $\frac{\overline{n-x-y} \times \overline{\dot{x}+\dot{y}}}{\sqrt{b^2 - \overline{n-x-y}^2}} = \frac{x\dot{x}}{\sqrt{a^2 - x^2}} - \frac{y\dot{y}}{\sqrt{d^2 - y^2}}$; wherein, by ſubſtituting for $\dot{x}$, its value found by the equation ariſing from the maximum, and dividing by $\dot{y}$, &c. we ſhall at length have $-FI + FI \times \frac{L \times FC \times BI}{K \times EI \times AF} = \frac{L}{K} \times \frac{BI \times EW}{EI} + \frac{BI \times EW}{EI}$, and therefore $FI \times FC \times BI \times L - FI \times EI \times AF \times K = BI \times EW \times AF \times K + BI \times EW \times AF \times L$; or, which is the ſame thing, $L \times \frac{FC \times CW - AF \times WE}{AF \times CE} = K \times \frac{BI \times EW + IE \times CW}{BI \times CE}$.

COROL. 1. $K : L :: \frac{\text{sin} < ACE}{\text{sin} < ACF} : \frac{\text{sin} < BEC}{\text{sin} < IEB}$; for $\frac{FC}{AC} = \text{sin} < ACW$, $\frac{AF}{AC}$ its cosine, $\frac{WE}{CE} = \text{sin} < WCE$, $\frac{CW}{CE}$ its cosine; all these to the radius 1, therefore $\frac{\text{sin} < ACE}{\text{sin} < ACF} = \frac{FC \times CW - AF \times WE}{AF \times CE}$; and, by the same way of reasoning, it may be shewn, that $\frac{\text{sin} < BEC}{\text{sin} < IEB}$ will be expressed by $\frac{BI \times EW + IE \times CW}{BI \times CE}$.

COROL. 2. Produce AC and BE, to G and H respectively, then K : L : : EG : CH. For the angle CHE is equal to the angle IEB, and the sine of the angle CEH is the same with that of its supplement BEC; therefore $\frac{\text{sin} < BEC}{\text{sin} < IEB} = \frac{CH}{CE}$, and likewise $\frac{\text{sin} < ACE}{\text{sin} < ACF} = \frac{EG}{CE}$; hence K : L : : $\frac{EG}{CE} : \frac{CH}{CE}$, that is, K : L : : EG : CH, which exactly agrees with the determinations in *page* 34.

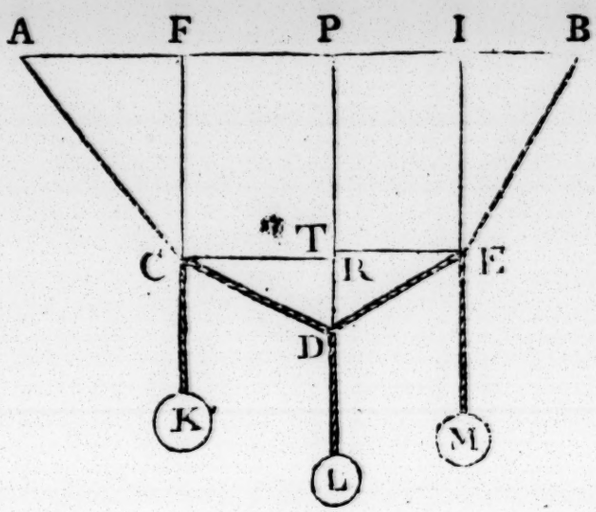
PROBLEM LIV.

LET three weights K, L, M, be freely suspended from the points C, D, E, of a perfectly flexible cord ACDEB, between the fixed points A, B, situated in the same horizontal line, to which the extremities of the cord are fastened; it is required to find the position of the string and weights, when sustained in equilibrio.

SOLUTION.

Let ACDEB be the required position. From the points C, D, E, draw to the horizontal line AB, the perpendiculars CF, DP, EI;

and from C and E, draw CR, ET, parallel to AB. Put AC $= a$, CD $= b$, DE $= c$, EB $= d$, AB $= n$, AF $= x$, BI $= y$, DR $= z$. Then FC $= \sqrt{a^2 - x^2}$, EI $= \sqrt{d^2 - y^2}$, FP $= $ CR $= \sqrt{b^2 - z^2}$, PD $= \sqrt{a^2 - x^2} + z$, and TD $= \sqrt{a^2 - x^2} + z - \sqrt{d^2 - y^2}$; conſequently TE, or its equal



PI $= \sqrt{c^2 + z - \sqrt{a^2 - x^2} - \sqrt{d^2 - y^2}}^2$, hence $\sqrt{c^2 - z - \sqrt{a^2 - x^2} - \sqrt{d^2 - y^2}}^2$ $= n - x - y - \sqrt{b^2 - z^2}$. Again, by Lemma 1, we have $\frac{K \times \sqrt{a^2 - x^2} + Lz + L \times \sqrt{a^2 - x^2} + M \times \sqrt{d^2 - y^2}}{K + L + M}$ for the perpendicular diſtance of the common center of gravity of the three weights K, L, M, from the horizontal line AB, which (becauſe a maximum) being put into fluxions, made equal to o, and properly reduced, gives $\dot{z} = \frac{\overline{K+L} \times x\dot{x}}{L \times \sqrt{a^2 - x^2}}$, ($y$ being conſtant) and $\dot{z} = \frac{M \times y\dot{y}}{L \times \sqrt{d^2 - y^2}}$ when x is conſtant. Now let the above equation, *viz.* $n - x - y - \sqrt{b^2 - z^2} = \sqrt{c^2 - z - \sqrt{a^2 - x^2} - \sqrt{d^2 - y^2}}^2$ be put into fluxions, firſt with y, and then with x conſtant, and then in the fluxional equations thence ariſing; ſubſtitute for $\dot{z}$, the reſpective values found above, we ſhall have, after due reduction being made, the following equations, $K \times TD \times AF \times CR = L \times FC \times CR \times TE - \overline{K+L} \times AF \times DR \times TE$, and $\overline{M+L} \times TD \times BI \times CR = TE \times CR \times EI \times L - DR \times TE \times BI \times M$. From theſe equations, it may be eaſily ſhewn, that K is to L,

as

as $\frac{\text{fin} \measuredangle ACD}{\text{fin} \measuredangle ACF \times \text{fin} \measuredangle FCD}$ to $\frac{\text{fin} \measuredangle CDE}{\text{fin} \measuredangle CDP \times \text{fin} \measuredangle EDP}$, and

that $L : M :: \frac{\text{fin} \measuredangle CDE}{\text{fin} \measuredangle CDP \times \text{fin} \measuredangle EDP} : \frac{\text{fin} \measuredangle DEB}{\text{fin} \measuredangle DEI \times \text{fin} \measuredangle BEI}$;

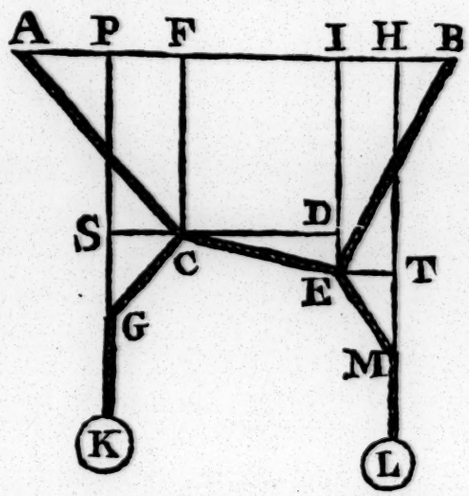
which exactly coincides with the concluſions in *pages* 93, 119, and 121.

PROBLEM LV.

LET the ſtring ACEB, whoſe ends A and B are faſtened to two tacks in the horizontal line AB, be divided into the given parts AC, CE, EB. Let alſo two weights K, L, act upon thoſe points of diviſion C, E, by means of the ſtrings CGK, EML, paſſing freely over two ſmall pullies fixed in the given points G, M; it is required to determine the poſition of the ſtrings, when theſe weights are ſuſtained in equilibrio.

SOLUTION.

Suppoſe it done, and ACGKLMEB the poſition required. From the points G, M, C, E, draw GP, MH, CF, EI, perpendiculars; and CS, CD, ET, parallels to AB. Put $AC = a$, $CE = b$, $EB = c$, $AB = n$, $AP = e$, $BH = d$, $PG = r$, $HM = s$, L and l for the reſpective lengths of the ſtrings CGK, EML, $AF = x$, $BI = y$; then will $FI = n - x - y$, $PF = SC = x - e$, $IH = ET = y - d$, $FC = \sqrt{a^2 - x^2}$, $ED = \sqrt{b^2 - n - x - y^2}$; hence we have $\sqrt{a^2 - x^2} + \sqrt{b^2 - n - x - y^2} = \sqrt{c^2 - y^2}$. Again, $GC = \sqrt{r - \sqrt{a^2 - x^2}^2 + x - e^2}$, and $EM =$



$\sqrt{\overline{s-\sqrt{e^2-y^2}}^2+\overline{y-d}^2}$; therefore, by Lemma 1, the perpendicular diſtance of the common center of gravity of the weights K, L, from the horizontal line AB, will be expreſſed $\frac{K}{K+L}$ into $r+L-\sqrt{\overline{r-\sqrt{a^2-x^2}}^2+\overline{x-c}^2}$ added to $\frac{L}{K+L}$ into $s+l-\sqrt{\overline{s-\sqrt{e^2-y^2}}^2+\overline{y-d}^2}$, which muſt be a maximum; and therefore in in fluxions we have $\frac{\dot{y}}{\dot{x}}=\frac{EI\times EM\times SG\times AF+SC\times FC\times EI\times EM}{-GC\times FC\times MT\times BI-GC\times FC\times ET\times EI}\times\frac{K}{L}$; and, by taking alſo the fluxion of the equation, expreſſing the relation of x to y, we have $\frac{\overline{n-x-y}\times\dot{x}+\overline{n-x-y}\times\dot{y}}{\sqrt{bb-\overline{n-x-y}^2}}=\frac{x\dot{x}}{\sqrt{a^2-x^2}}-\frac{y\dot{y}}{\sqrt{e^2-y^2}}$, that is, $\frac{FI\times\dot{x}}{DE}+\frac{FI\times\dot{y}}{DE}=\frac{AF\times\dot{x}}{FC}-\frac{BI\times\dot{y}}{EI}$, and conſequently $\frac{\dot{y}}{\dot{x}}=\frac{FI\times FC\times EI-ED\times AF\times EI}{-FC\times FI\times EI-DE\times BI\times FC}$, hence $L\times\frac{FI\times FC\times EI-ED\times AF\times EI}{-FI\times EI-DE\times BI}=K\times\frac{EI\times EM\times SG\times AF+SC\times FC\times EI\times EM}{-GC\times MT\times BI-GC\times ET\times EI}$.

Corol. 1. If $ED=0$, or the line EC parallel to the horizon, the laſt equation becomes $L\times FC=K\times\frac{EM\times SG\times AF+SC\times FC\times EM}{GC\times MT\times BI+GC\times ET\times FC}$.

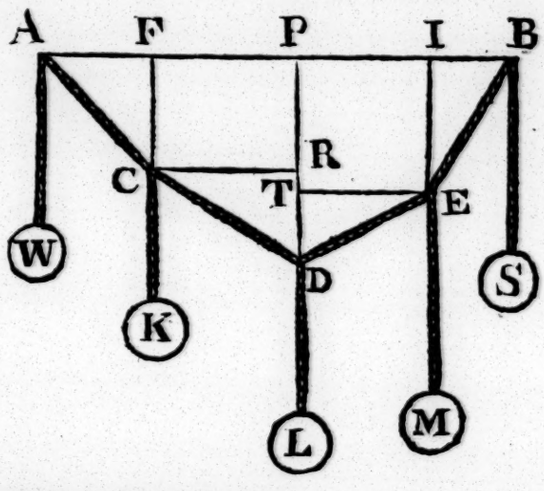
Corol. 2. If SC and ET are each $=0$, our general equation becomes $L=K\times\frac{AF\times FI\times EI+AF\times DE\times BI}{ED\times AF\times BI+BI\times FI\times FC}$; wherein, if we ſuppoſe alſo $ED=0$, then $L=K\times\frac{AF}{BI}$, and $L:K::AF:BI$.

PROBLEM LVI.

THE ſtring WACDEBS ſlides freely over two ſmall pullies placed at the points A, B, in the horizontal line AB; at the extremities of this line, the weights W and S are appended, as are alſo the weights K, L, M, at the aſſigned points C, D, E; it is required to find the poſition of the ſtring, when the weights W, S, ſuſtain the others K, L, M, in equilibrio.

SOLUTION.

Let ACDEB be the required poſition of the ſtring; and from the points C, D, E, draw DF, DP, EI, perpendicular to AB; draw alſo CR and ET parallel to AB. Put $AB = n$, $CD = b$, $DE = c$, $AF = x$, $BI = y$, $FP = u$, $AC = z$, L and l for the reſpective lengths of the ſtrings CAW, EBS; then will $FC = PR = \sqrt{z^2 - x^2}$, $DR = \sqrt{b^2 - u^2}$, $DP = \sqrt{z^2 - x^2} + \sqrt{b^2 - u^2}$, whence we get $BE = \sqrt{\overline{\sqrt{z^2 - x^2} + \sqrt{b^2 - u^2} - \sqrt{c^2 - \overline{n - x - y - u}^2}}^2 + y^2}$. Hence, by Lemma 1, we have $\frac{1}{W + K + L + M + S}$ into $LW - Wz + K \times \sqrt{z^2 - x^2} + L \times \sqrt{z^2 - x^2} + L \times \sqrt{b^2 - u^2} + M \times \sqrt{z^2 - x^2} + M \times \sqrt{b^2 - u^2} - M \times \sqrt{c^2 - \overline{n - x - y - u}^2} + l \times S - S \times \sqrt{\overline{\sqrt{z^2 - x^2} + \sqrt{b^2 - u^2} - \sqrt{c^2 - \overline{n - x - y - u}^2}}^2 + y^2}$ for the perpendicular diſtance of the common center of gravity of all the weights, from the horizontal line AB, which muſt be the greateſt poſſible. In fluxions,



with z variable, gives $K \times \frac{AC}{FC} + L \times \frac{AC}{FC} + M \times \frac{AC}{FC} = S \times \frac{EI \times AC}{FC \times BE} + W$; and by taking the fluxion of the ſame expreſſion, with x variable, we ſhall have $K \times \frac{AF}{FC} + L \times \frac{AF}{FC} + M \times \frac{AF}{FC} = \frac{S \times EI \times AF}{FC \times BE} + \frac{S \times P \times EI}{DT \times BE} - M \times \frac{PI}{DT}$; and again, by taking the fluxion of the above expreſſion, firſt with y, and then with u variable, there will ariſe the following equation, *viz.* $\frac{S \times PI \times EI}{BE \times DT} - M \times \frac{PI}{DT} = S \times \frac{BI}{BE}$, and $L \times \frac{CR}{DR} + W \times \frac{CR}{DR} = \frac{S \times CR \times EI}{BE \times DR} + \frac{S \times PI \times EI}{DT \times BE} - M \times \frac{PI}{DT}$. If for $-M \times \frac{PI}{DT}$ in the ſecond of the foregoing equations, we ſubſtitute its value, found by the third equation, we ſhall have $\overline{K+L+M} \times \frac{AF}{FC} = \frac{S \times BI}{BE} + \frac{S \times EI \times AF}{FC \times BE}$, which is alſo equal to $\frac{S \times AF \times EI}{FC \times BE} + \frac{AF \times W}{AC}$; and therefore $AF \times BE \times W = S \times BI \times AC$, or $W : S :: \frac{BI}{BE} : \frac{AF}{AC}$. This ratio of W to S might have been otherwiſe determined, as in the following manner:

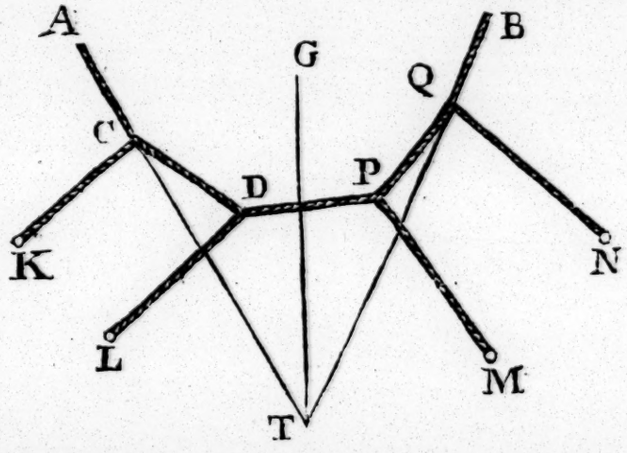
Let the forces acting in the directions CD, DE, be repreſented by m and n reſpectively; then, becauſe W is the force acting in the direction AC, it will be $W : m :: \text{ſin} < FCD : \text{ſin} < FCA$; and again, $m : n :: \text{ſin} < PDE : \text{ſin} < PDC$, alſo $n : S :: \text{ſin IEB ſin} < IED$, by Problem III, therefore $W : S : \text{ſin} < BEI : \text{ſin} < ACF$; for the ſines of the angles FCD, PDC, are equal, as are alſo thoſe of the angles PDE, IED. If the weights K, L, M, act in oblique directions, the ratio of W to S will be that of the reſpective products of the ſines of the angles, (the former of the ſines of the angles towards the right-hand, and the

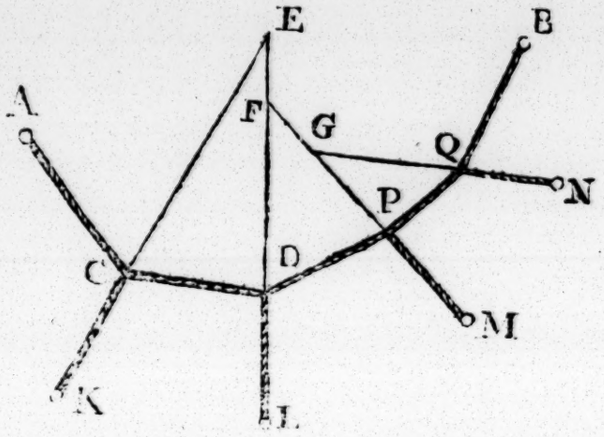
the latter of thoſe of the angles towards the left-hand) which thoſe directions make with the correſponding parts of the ſtring.

COROL. 1. $W : S :: \frac{BI}{BE} : \frac{AF}{AC}$, or $: : \text{ſin} < BEI : \text{ſin} < ACF$, let the number of intermediate weights (acting in directions perpendicular to the horizon) be what they will.

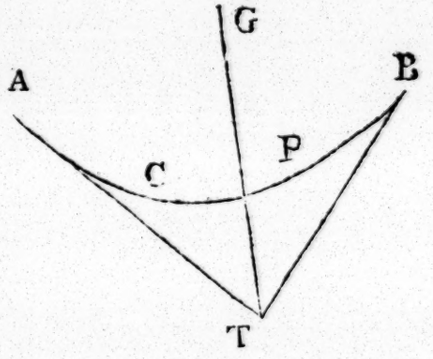
COROL. 2. $K + L + M = S$ into $\frac{EI}{BE} + \frac{FC}{AF} \times \frac{BI}{BE}$.

COROL. 3. If the firſt ſide AC, and laſt BQ, of any funicular polygon ACDPQB, formed by the forces K, L, M, *&c.* acting in the directions KC, LD, *&c.* be produced until they interſect, as in T; and the angle ATB divided by the right line TG; ſo that the ſines of the angles ATG, GTB, may be reciprocally as the forces A and B, acting in the directions AC, BQ; then will TG be the direction of the effort, ariſing from the joint action of all the intermediate forces K, L, M, *&c.* and will be to the reſiſtance of either of the fixed points A, B, as the ſine of the total angle ATB, to that of the correſponding partial angle ATG or BTG; which agrees with what M. *Bernoulli* has inveſtigated, by a different method, in his *Eſſay de la Manœuvre des Vaiſſeaux*, chap. 15, prop. 3.

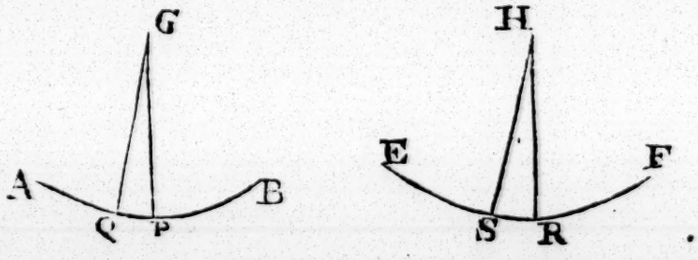




COROL. 4. If the directions of the reſpective forces be produced to their points of interſection E, F, G, *&c.* and *e*, *f*, *g*, &c. be put for the forces with which the parts CD, DP, PQ, *&c.* are drawn in the directions of their lengths, then will $K = e \times \frac{DC}{CE}$, $L = f \times \frac{DP}{FD}$, $M = g \times \frac{PQ}{PG}$, *&c.*



COROL. 5. If the polygon ACDPB degenerates into a curve AB, by the action of innumerable forces, applied to the ſeveral parts of the ſtring, or by the abſolute weights of the ſaid parts; then (having from the points of ſuſpenſion A, B, drawn the tangents AT, BT) if the forces act upon this cord or ſtring, in directions perperpendicular to its curvature in the ſeveral points thereof, the line GT, which biſects the angle ATB, will be the direction of the joint effort of all the forces acting upon the propoſed curve. But if thoſe forces act in directions perpendicular to the horizon, which is the caſe when the curve is formed by the gravitation of its parts; then GT, perpendicular to the horizon, will become the direction of the joint effort of all the forces acting upon the ſaid curve.



COROL. 6. Let AB and EF be two curves, formed by an infinite number of forces, acting in directions perpendicular to the ſeveral parts of thoſe curves; and let alſo GP and GQ, HR and HS, be radii of curvature to the points P, Q, and

R, S. This premiſed, let M and T repreſent the forces acting in the directions PG, HR, g and h thoſe with which the indefinitely ſmall parts QP, SR, are drawn in directions of their lengths; then, if we ſuppoſe the forces MT in equilibrio with the others g, h, there will ariſe $\frac{g \times PQ \times T}{GP} = \frac{h \times RS \times M}{HR}$, becauſe, by Corol. 3, $M = \frac{g \times PQ}{GP}$, $T = \frac{h \times RS}{HR}$, hence $g : h :: \frac{GP \times M}{PQ} : \frac{HR \times T}{RS}$; wherein, if we take PQ = RS, g will be to h, as GP × M to HR × T.

FINIS.

ERRATA.

PAGE 8, Line 17, *for* E^2, *read* E. P. 36, *for* N upon the fig. *read* A; and, *for* maximum, at l. 3, from the bottom, *read* minimum. P. 55, *for* EH, in the laſt line, *read* FH. P. 59, l. 13, *for* G, *read* GD. P. 75, l. 11, *for* AE, *read* AF. P. 100, l. 16, *dele* c.

The following BOOKS are all written by Mr. THOMAS SIMPSON, F. R. S. and printed for J. NOURSE.

I. ESSAYS ON SEVERAL CURIOUS AND USEFUL SUBJECTS, IN SPECULATIVE AND MIXED MATHEMATICKS; in which the moſt difficult Problems of the firſt and ſecond Books of *Newton*'s *Principia* are explained; in 4to.

II. MATHEMATICAL DISSERTATIONS on a Variety of Phyſical and Analytical Subjects; in 4to.

III. MISCELLANEOUS TRACTS on ſome curious and very intereſting Subjects in Mechanics, Phyſical-Aſtronomy, and Speculative Mathematicks, in 4to.

IV. THE DOCTRINE OF ANNUITIES AND REVERSIONS, deduced from general and evident Principles; with uſeful Tables, ſhewing the Values of ſingle and joint Lives, *&c.* in 8vo.

V. A TREATISE OF ALGEBRA; wherein the fundamental Principles are fully and clearly demonſtrated, and applied to the Solution of a great Variety of Problems, and to a Number of other uſeful Enquiries. 2d Edition, in 8vo.

VI. THE DOCTRINE AND APPLICATION OF FLUXIONS; containing (beſides what is common on the Subject) a Number of new Improvements in the Theory, and the Solution of a Variety of new and very intereſting Problems in different Branches of the Mathematicks, 2 vols. 8vo.

VII. TRIGONOMETRY, PLAIN AND SPHERICAL, with the Conſtruction and Application of Logarithms, in 8vo.

VIII. SELECT EXERCISES for Young Proficients in the Mathematicks; containing, beſides a choice Collection of Problems, both algebraical and geometrical, the whole Theory of Gunnery; a very accurate and ſuccinct Demonſtration of the firſt Principles of Fluxions; and a Set of Tables for the Valuation of Annuities and Reverſions, more comprehenſive than any extant. 8vo.

Where may be had,

I. A TREATISE, containing the ELEMENTARY PART of FORTIFICATION, Regular and Irregular, with Remarks on the Conſtructions of the moſt celebrated Authors, particularly of Marſhal DE VAUBAN and Baron COEHORN; in which the Perfection and Imperfection of their ſeveral Works are conſidered. For the Uſe of the Royal Academy of the Artillery at Woolwich. Illuſtrated with thirty-four Copper-plates; by JOHN MULLER, 8vo.

II. A TREATISE OF ALGEBRA, in three Parts; containing, 1. The Fundamental Rules and Operations. 2. The Compoſition and Reſolution of Equations of all Degrees, and the different Affection of the Roots. 3. The Application of Algebra and Geometry to each other. To which is added, an Appendix, concerning the general Properties of Geometrical Lines. By COLIN MACLAURIN, M.A. late Profeſſor of the Mathematicks in the Univerſity of Edinburgh, and F. R. S. in 8vo.

7.1559
360
21.46770
2
429 = 20²

31.194 — 11.781
6

(b) 19.413 = R²

3.235
2146.7
22645
13910
12940
9235
6470
20745.45
12

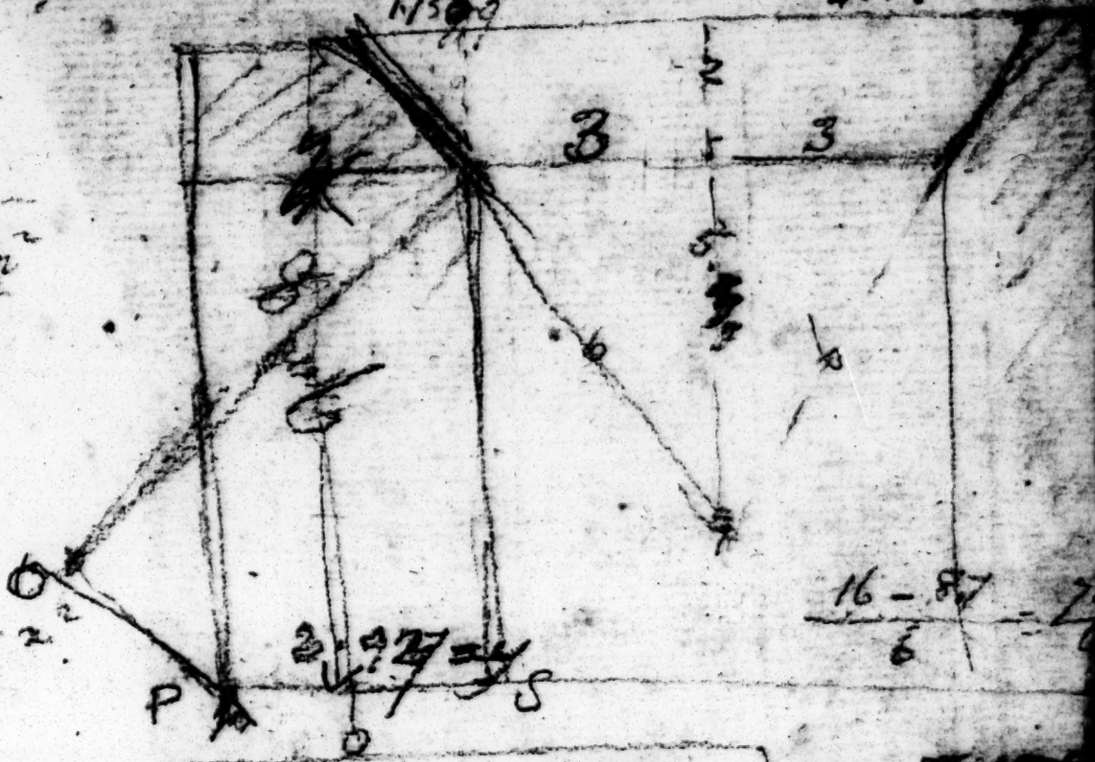
$\frac{1}{2}PS = \frac{1}{2}$

21204
4240
63612
7068
138804

$\frac{1}{2}$

$\frac{1}{2}$

$\frac{1}{2}$


3
3
P
Q
S